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Demand Models, Revenue Curves and Profit

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In this short paper some common conventions regarding revenue curves are questioned and revenue is contrasted with profit. It turned out that obeying the law of demand – non-increasing demand function, is insufficient to characterize the revenue curve. Non-increasing demand function may result in increasing and/or decreasing revenue curves, concave and/or convex revenue curve and even curves with multiple local extreme points. Fortunately, a sufficient condition is found, which enables to better characterize the revenue curve. Based on this result, it is shown that the quantity that maximizes the profit differs from the quantity that maximizes the revenue. Further, the difference can be substantial and the profit is more sensitive to quantity changes than the revenue.

Keywords: *demand, revenue, profit, firms*

Introduction

Economists "think of the economic system as being coordinated by the price mechanism" (Coase 1937). This price mechanism associates quantities with prices and is stabilized at a price in which an equilibrium is reached between demand and supply. Revenue is the multiplication of the price times the quantity sold and is expressed as a function of the price. The provision of products, including services, involves costs. Consequently, a necessary condition for the survival of a provider, in the long run, is that the total cost does not exceed the revenue. Of course, providers aim not at survival but at profit maximization – maximizing the difference between the revenue and the total cost. Firms are providers, of products and/or services, and as Coase (1937) noted "the distinguishing mark of the firm is the supersession of the price mechanism." This is because: 1) the price mechanism does not account for the costs and 2) the costs are independent of the retail price. Nevertheless, firms, and providers in general are affected by the price mechanism through the revenue. Accordingly, there is a need to integrate firms' coordination with that of the economic system. In previous studies, either specific demand functions are assumed or certain assumptions are made about the revenue curve without specifying a demand function. In many studies the profit is maximized but in other studies the revenue is maximized. In this note the revenue curve and its relationship to the demand model are examined. Some surprising observations lead to a more careful definition.

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Other central issues are the schedule of the decisions and the decision variables. The decision variables are prices and quantities – providers should decide how much to offer for sale and at what price. However, there is often a long-time leg between the decisions; e.g., when products are shipped by sea transport from overseas, or, and even to a larger extent, when infrastructure has to be built or expanded. Further, the selection of quantities often has a substantial ramification, regarding costs, in particular. Prices are set long after, when the quantities and the consequences of their selection can hardly be changed, if at all. Hence, quantities' determinations are no less significant and there is a need to understand their implications.

The paper is organized as follows. In the next section, relevant literature is briefly reviewed. Then, the revenue-demand relationships are examined and discussed, profit is considered in the fourth section and conclusions a direction for future research are offered in the last section.

Literature Review

As noted, the revenue, which is a major determinant of providers' profit, connects providers with the economic system. The price mechanism coordinates the economic system by associating the demanded quantity with the price, whose product: price times the corresponding demand, is the revenue. The simplest and most common approach to model the revenue is to assume a specific demand model and the most popular demand model is a linear function. So popular that one wondered: "I never understood as a first-year student why we called the demand curve a "curve" when it was a straight line"[1]. The popularity of the linear demand function in the literature is also prominent; e.g., in Crockett (2013), in the surveys of Huang et al. (2013), Aust and Buscher (2014) and Kumar et al. (2016) and in more recent studies, e.g., Huang et al. (2016), Duan and Ventura (2020), Bos and Vermeulen (2021), Hauck et al. (2021) and Li and Liu (2021). The linear model is popular because it is simple. Choi (1991) noticed the difficulty to derive analytical results with nonlinear models. Similarly, Huang et al. (2013) explain this use of the linear models, because it gives rise to explicit results and it is relatively easy to estimate the parameters. Oddly enough, Desiraju and Moorthy (1997) drew a non-linear function in their introduction, but diverted to a linear function in the analysis that follows. However, Choi (1991) showed that many results reverse when the linear demand is replaced by nonlinear functions, while Lau and Lau (2003) showed that slight changes in the demand curve could lead to significant changes in optimal solutions. Further, Huang et al. (2013) noted that in most practical cases the assumption of a linear demand function does not correspond to reality. Hence, in this study, the linear function is avoided. Other

[1]https://econ101help.com/why-is-the-demand-curve-a-straight-line/.

models include the power and exponential functions. Duan and Ventura (2020) criticized the linear, power and exponential demand functions and proposed the logit function as an alternative.

Another approach is to make certain assumptions about the revenue curve without specifying a demand function. Zusman and Etgar (1981) assumed that, "in the range of the analysis", the revenue curve's first derivative is positive while the second derivative is negative. Namely, the curve is increasing and concave. Besbes and Zeevi (2009) assumed that the revenue function is concave. While this approach enables profit maximization, it is often used to maximize revenue only, e.g., Besbes and Zeevi (2009, 2015).

In the next section, the concavity assumption of the revenue curve and its aptness to popular demand model are examined and in the section that follows, the profit function is examined.

Demand and Revenue

The *law of demand* (Marshal 1892), namely, the demand $D(p)$ is non-increasing in the price, p, is a convention. In addition, it is explicitly assumed that both the price and the demand are non-negative. When it comes to the revenue curve, concavity is commonly assumed. To a lesser extent, but still common is the assumption that the demand is an invertible function of the price. While Besbes and Zeevi (2009) proclaimed that "These assumptions are quite standard in the revenue management literature …", their validity, concavity of the revenue curve, in particular, are still relevant questions.

Consider, first, the second most popular demand function, the power model: $D(p) = \beta \cdot p^{-\alpha}$, where both α and β are strictly positive. The corresponding revenue curve is: $R(p) = p \cdot \beta \cdot p^{-\alpha} = \beta \cdot p^{1-\alpha}$. If $\alpha < 1$, $R(p)$ is monotonically increasing in p and concave, while if $\alpha = 1$, $R(p) = \beta$, the R(p) independent of p. However, if $\alpha > 1$, $R(p)$ is monotonically decreasing in p and convex.

Figure 1. *Revenue Curve with Exponential Demand, $\alpha = 0.95$, $\beta = 100$*

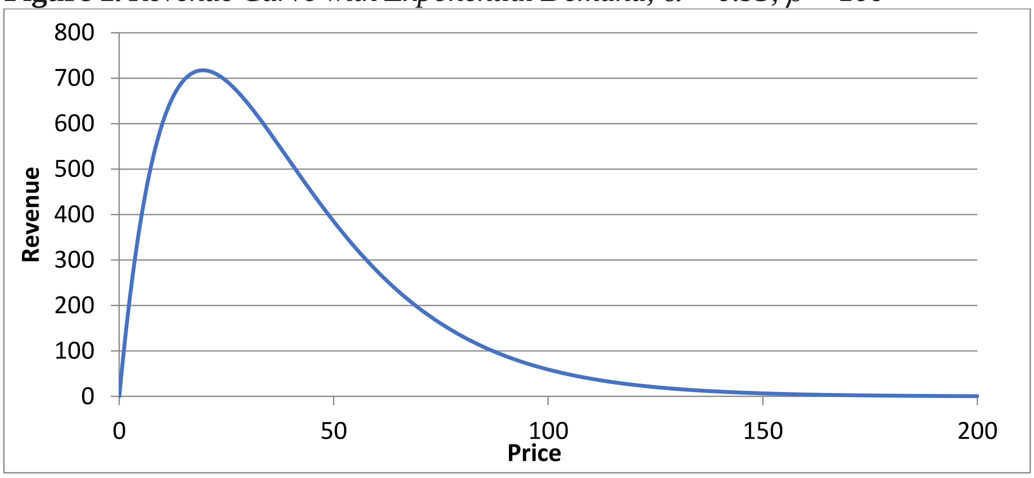
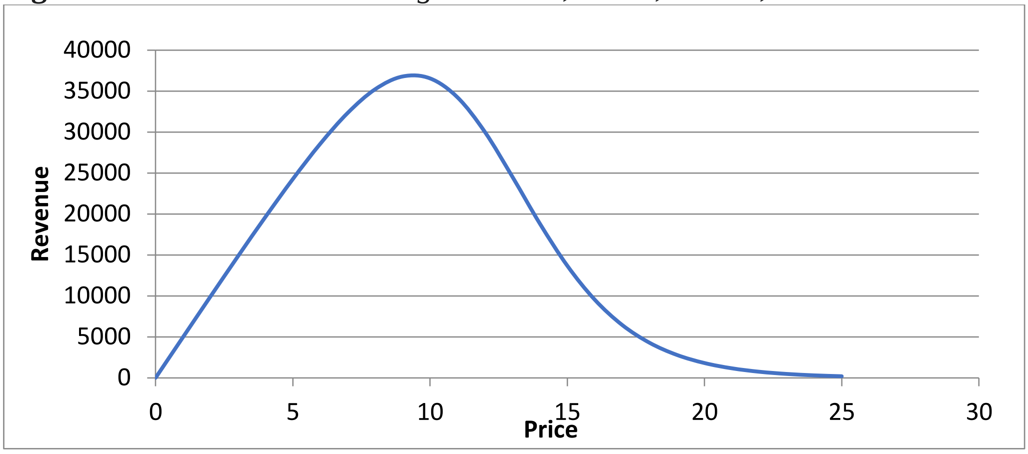


Figure 2. *Revenue Curve with Logit Demand, a = -6, b = 0.5, C = 5000*



The next two most popular demand function are the exponential function: $D(p) = \beta\cdot\alpha^p$, where $\beta > 0$ and $0 < \alpha < 1$, and the logit function: $D(p) = C\frac{e^{-(a+bp)}}{1+e^{-(a+bp)}}$, where $a < -2$, $b > 0$, and $C > 0$ denotes the market size. Both obey the law of demand and examples of corresponding revenue curves are displayed in Figures 1 and 2. Clearly, the revenue curves are only partially concave. This may not affect the results of previous studies because both curves are uni-modal and concave up to the maximal point and to some range after.

Next, consider the function presented in Table 1 and Figure 3. This is a piecewise linear function – the values in Table 1 are the breakpoints of the curve, which are connected by straight lines. This function certainly obeys the law of demand and is invertible, too. However, the revenue curve of a linear demand function is a quadratic function. Consequently, the revenue function, which corresponds to Table 1 and Figure 3, is the upper envelop of a series of quadratic functions and is not uni-modal, but has a single global maxima and few local maxima, as can be seen in Figure 4.

Table 1. *Piecewise Linear Demand Function*

P	0	2	4	5	6	8	12	20
D(*p*)	28	12	6	5	4	3	2	1

Figure 3. *Piecewise Linear Demand Function*

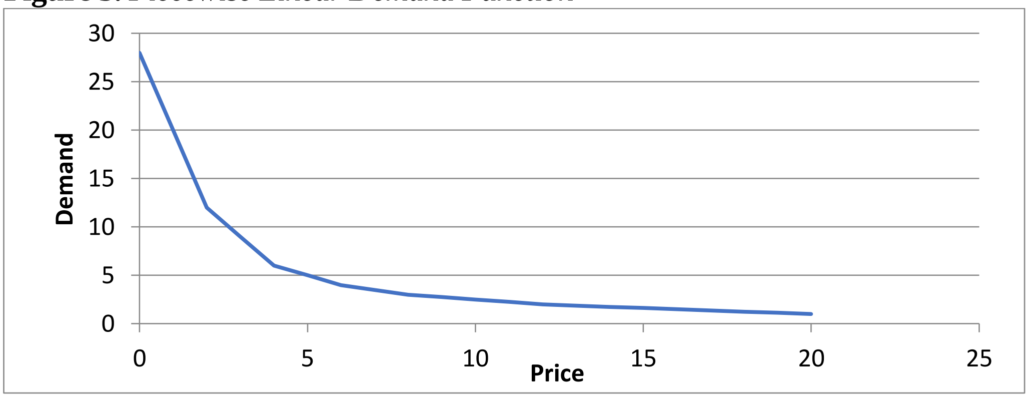
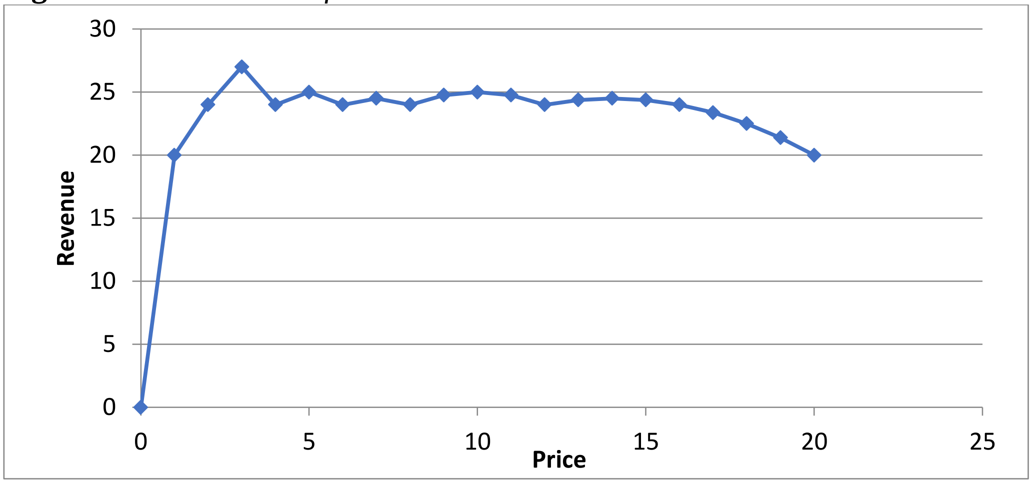


Figure 4. *Revenue Curve for the Piecewise Linear Demand Function*



In sum, thus far, the revenue curve may be neither increasing nor concave and not even monotone. Evidently, a non-increasing and invertible demand function is insufficient to fully characterize the revenue curve. Nevertheless, the following observation does hold.

Proposition 1: When the demand function is non-increasing in p, the revenue curve, $R(p)$, cannot be both increasing and convex in p.

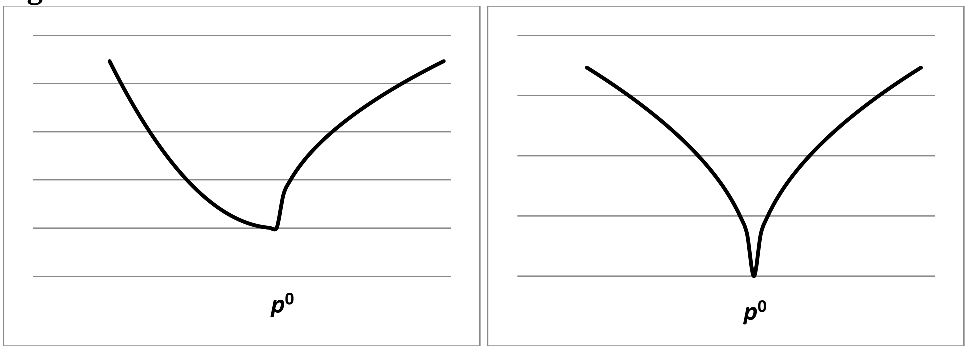
Proof: Suppose $R(p)$ increases for some p^0, that is $R(p^0 + \varepsilon) > R(p^0)$ for some ε > 0. Since $D(p)$ is non-increasing in p: $R(p^0 + \varepsilon) = (p^0 + \varepsilon) \cdot D(p^0 + \varepsilon) \leq (p^0 + \varepsilon) \cdot D(p^0)$. Hence, the increase in $R(p)$ is no bigger than a linear increase at rate $D(p^0)$.

Further, the quote from Besbes and Zeevi (2009) in the beginning of this section is incomplete. The complete statement is: "These assumptions are quite standard in the revenue management literature, resulting in the term *regular* affixed to demand functions satisfying these conditions; see, e.g., Talluri and van Ryzin (2005, §7)." However, the list of regularity assumptions in (Talluri and van Ryzin 2005) is longer. In particular, the first assumption is that the demand function is continuously differentiable. With this addition, the following observations hold, too.

Corollary 1: When the demand function is continuously differentiable and non-increasing in p, once $R(p)$ decreases for some p^0, it decreases for any p > p^0.

Proof: Suppose $R(p)$ decreases up to p^0 and increases from p^0 onward. Then, by proposition 1, $R(p)$ is non-convex from p^0 onwards – see Figure 5. Consequently, $R(p)$ is not differentiable at p^0, contradicting the assumption that $\partial R(p)/\partial p$ is well defined for any value of $p \geq 0$, including p^0 – the breakpoint in both parts of Figure 5.

Figure 5. *Break Points*



Corollary 2: When the demand function is continuously differentiable and non-increasing in p, there exists a single point, p^R, where the revenue, $R(p)$, is maximized.

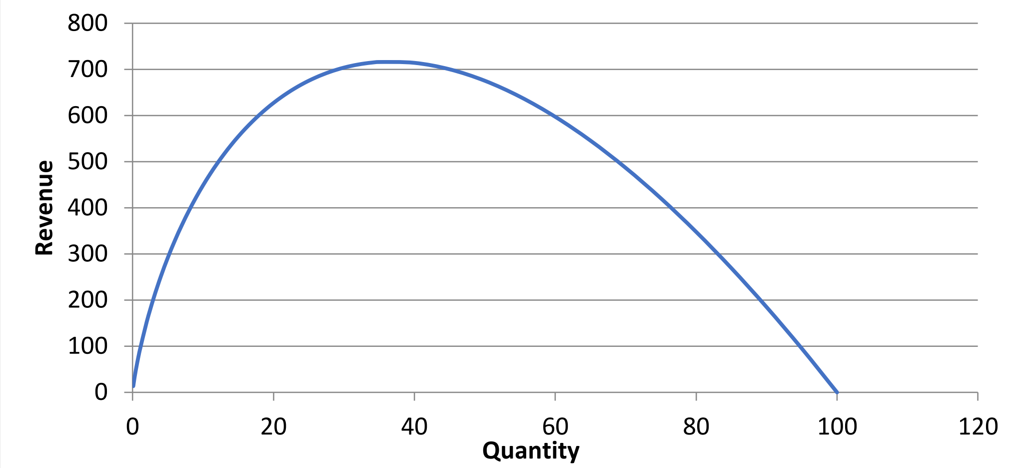
The proof of corollary 2 is straightforward and hence omitted.

Functions which are not continuously differentiable are piecewise linear functions; e.g., Table 1 and Figure 3. This shows the significance of the continuously differentiable assumption, which might be considered restrictive. However, it is only a necessary condition and other conditions might do, too.

Profit vs. Revenue

The economic system aims at equilibrium, while firms, as noted, aim at profit maximization. Profits equals total revenues minus total costs, but the costs depend on the quantity, not on the sale price. In addition, the demand is exogenous and firms have a limited control on its value. As noted by Zusman and Etgar (1981), in real world situations the quantity is always monitored, while the demand is seldomly fully monitored. Indeed, what providers, e.g., firms, always do is to determine the amount, Q, which is produced and offered for sale. Since there is one to one relationship between prices and quantities, the quantity may be legitimately regarded as the control variable. Namely, $R(Q) = Q \cdot p(Q)$. It then follows that $p(Q)$ is also continuously differentiable and non-increasing in Q, and proposition 1, corollary 1 and corollary 2 hold for $R(Q)$ as well. The revenue curve, $R(Q)$, which corresponds to Figure 1 is shown in Figure 6, for example.

Figure 6. *Revenue Curve R(Q) for $D = \beta{\cdot}\alpha^p$, with $\alpha = 0.95$*



It should be understood, however, that $R(Q)$ is an upper-bound on the revenue. As noted, the demand is seldom monitored. Therefore, the right price for the quantity offered is not known. Whether the price is too high or too low, the actual revenue will be smaller. Hence, lower revenue is the likely result.

The profit $\Pi(Q) = R(Q) - C(Q)$, and it is assumed that the total costs, $C(Q)$, is non-negative and non-decreasing in Q. Then, either one of the following cases may occur:

1. The cost curve lies above or on the revenue curve. This implies non-positive profit: $\Pi(Q) \leq 0$, for all $Q > 0$.
2. There exist a quantity $Q' \geq 0$ for which the cost is smaller than the revenue: $C(Q') < R(Q')$, and hence, $\Pi(Q') > 0$.

Rationality dictates positive quantity only when positive profit can be expected; i.e., case 2: there exists $Q > 0$ for which $R(Q) > C(Q)$. Adding an assumption about the cost function leads to the following observation.

Proposition 2: When $C(Q)$ is non-negative, non-decreasing and convex in Q and $R(Q)$ is uni-modal with a maximum point at $Q^R > 0$, the maximum profit is obtained at a quantity $Q^\Pi \leq Q^R$.
Proof: $R(Q)$ is monotonically increasing and concave, at least for $Q < Q^R$. Thus, the convexity of $C(Q)$ implies that it may intersects $R(Q)$ at most twice and the profit is positive between the intersection points: $Q^1 < Q < Q^2$, and non-positive elsewhere. Further, the concavity of $R(Q)$ implies that $R' = dR(Q)/dQ$ is non-increasing in Q**.** The maximum of $R(Q)$ is obtained at Q^R where $R' = 0$, while that of the profit is obtained at Q^Π where $R' - dC(Q)/dQ = 0$. The results then follow since $dC(Q)/dQ \geq 0$ because $C(Q)$ is non-decreasing.

While both components of the profit are considered as functions of the quantity, they are highly independent. An implication of this independence is that changing the cost function changes only Q^{Π}, not Q^{R}. To illustrate, consider Figure 6 where $Q^{R} \approx 35$ and suppose the cost function is linear. When the variable cost is \$5/unit, $Q^{\Pi} \approx 28$, while when the variable cost is \$8/unit, $Q^{\Pi} \approx 24$. Moreover, Q^{Π} is 20% less than Q^{R}, in the first case, and more than 30% less in the second case. The revenue associated with Q^{Π} is about 3% less than the maximum in the first case and about 6.8% less in the second case. However, while the profit associated with Q^{R}, is about 2.7% less than the maximum in the first case – similar to the revenue change, the difference grows to about 9.25% in the second case. These examples indicate that the difference between Q^{Π} and Q^{R}, that is between revenue and profit maximization can be substantial and that the profit is more sensitive to quantity changes than the revenue.

Summary

In this note, the relationship between the demand model and the revenue curve has been examined and then, revenue was contrasted with profit. It has been found that more caution is needed when the revenue curve is considered. The curve is not necessarily concave, it might be decreasing, and might even have multiple local extreme points. However, when the demand function is continuously differentiable and non-increasing in the price, then if the revenue curve is not monotonically decreasing, it is first increasing and concave. Based on the last observation, it was shown that the quantity that maximizes the profit differs from the quantity that maximizes the revenue. Further, the difference can be substantial and the profit is more sensitive to quantity changes than the revenue.

An implication of these results is that accurate estimations and forecast of both the revenue and the cost curves are required in order to maximize the profit. A primary prerequisite with regard to the revenue curve is to relax, or to give up the assumption that the demand function is known. In general, the quantity sold is the minimum between the quantity offered to the demand, but the real situation is more complicated. A quantity is determined first and then, a price is set. If the price is too high, a surplus will be created. Often, a price discount can be offered and then additional units are sold, but then, different prices determine the revenue. If, on the other hand, the price is too low, the whole quantity will be sold and a shortage might be sensed. Making additional supply is harder than altering prices and takes time, but can still be done, in some cases, and might be accompanied with price increase. Then, again, the revenue calculation is affected. Accurate modeling of the revenue and cost curves are challenges which are left for future work.

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