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Gregory T. Papanikos, President, ATINER.

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Important Dates

- Abstract Submission: **18 April 2023**
- Acceptance of Abstract: 4 Weeks after Submission
- Submission of Paper: **19 June 2023**

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- Athens Sightseeing: Old and New-An Educational Urban Walk
- Social Dinner
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Relationship between Mathematics Teachers' Teaching Styles and Students' Achievement in Mathematics

By Angelina P. Lumbre[*], Ma. Nympha Beltran-Joaquin[±] & Sheryl Lyn C. Monterola[°]

This study investigated the relationship between teaching styles of 30 grade 9 mathematics teachers and the achievement of their 1489 students. The Grasha Model of learning styles was adapted in the study. Results of the analysis indicated a significant relation between teaching style and student achievement based on students' highest mean percentage score. Approximately 39% of the total variance in students' achievement is attributable to the difference between the teaching styles. Post hoc comparisons showed that students whose teachers exhibit the Expert style, as well as those whose teachers use a combination of teaching styles have significantly higher achievement scores than students whose teachers employ the Formal Authority style of teaching. Since favorable teaching styles were identified from this study among grade 9 teachers and students, a wider research on the association of teaching styles and student achievement in mathematics focusing on other grade levels is being put forward. This may also help determine at which grade level student achievement starts to decline and further identify effective teaching styles appropriate for each grade level. Pre-service teacher training and in-service teacher retooling may likewise be conducted to leverage academic learning by allowing teachers to discover their teaching styles and improve on them.

Keywords: *teaching style, students' achievement, mathematics*

Introduction

Mathematics plays a vital role in one's daily life. Concepts and skills learned in mathematics allow one to think analytically and critically leading to informed decisions. In addition, everyday activities such as driving, cooking, playing games, project planning, and many others require specific mathematical understanding and application. As such, the Department of Education (2016) emphasizes the importance of using experiential and situated learning principles in its K to 12 Basic Education Curriculum for Mathematics. These learning principles allow Filipino learners to make sense of their direct everyday experiences where they can recognize the use and relevance of mathematics (Department of Education 2016).

However, studies on Filipino students' academic performance reveal that many students have an insufficient level of achievement in mathematics. In particular, the study of Capate and Lapinid (2015) on the performance and

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difficulties of students during the first year of implementation of the K to 12 Mathematics showed that students' level of achievement in Mathematics was still at the beginning to developing stages. Students were also found to have lacked critical problem-solving skills as they had not been re-checking their answers and had been incorrectly applying formulas, properties, theorems, or laws. The 2018 and 2019 National Achievement tests (NAT) attest to Filipino learners' deficient performance, especially in math (Malipot, 2019, Department of Education 2019b). Critical thinking skills registered the lowest mean in mathematics subjects.

Moreover, in their first attempt to join the Program for International Student Assessment (PISA) of the Organization for Economic Co-operation and Development (OECD) in 2018, Filipino students were found to be performing below Level 1 as defined by PISA's six proficiency levels (Department of Education 2016). As a result, the Philippines ranked least among the 79 participating countries in PISA 2018.

Level 1 in PISA's six proficiency levels is when students can answer basic concepts, carry out routine problems, and perform basic operations. This level is far from what is expected of them at Level 3 based on OECD average scores. Level 3 is when students can directly apply basic problem-solving strategies, interpret representations, and reason. In addition, the results of the 2018 PISA show that 19.7% of Filipino examinees attained proficiency levels 2 to 4. These levels indicate skills where students can employ basic algorithms, formulae, procedures, or conventions to solve problems involving whole numbers. Further, it is sad to note that only 0.01% of students performed within proficiency levels 5 to 6, where a learner is expected to develop and work with models for complex situations, identifying constraints and specifying assumptions.

Accordingly, with the similar result of the NAT and PISA reported, and although DepEd officials are already aware of what the result of the test will be, given the performance of students in NAT in the previous years, joining the PISA opened an opportunity for the education experts to assess where the Filipino learners currently are and go on with its call to quality education for all learners (Malipot 2019). Furthermore, this phenomenon calls for the need to find a way to make students learn mathematics to achieve better in the subject.

However, most of the literature focused only on student factors affecting achievement. Teacher factors affecting student achievement in mathematics may open another discussion that will help improve the quality of mathematics education in the country. For example, Liwag (2008) recommended observing different teaching strategies in mathematics classrooms, which may uncover more causes of poor mathematics reasoning abilities among students. In addition, Paz (2009), in his studies about teaching styles and students' performance, recommended that future researchers do more research on teaching styles in other fields, such as mathematics. Further, exploring teachers' role in the classroom instead of using research-based interventions is more recommended as these interventions are not fully implemented due to their complexity and inaccessibility, especially in regular public high schools.

Hence, looking into teacher factors that affect students' achievement may help the Department of Education's continuous call for quality education. For instance, research on teaching styles and mathematics instruction has established that teachers' mastery of content and teaching style positively influence students' mathematics performance. Furthermore, Caluya (2000) found a significant relationship between students' mathematics thinking and teachers' transformational practices in mathematics. Moreover, it was revealed that certain traits and practices of teachers influenced the students' demonstration of thinking skills. Sasing (2014), in her research on the influence and relationship between teacher's mathematical pedagogical knowledge (MPK), mathematical disposition (MD), and mathematics teaching experience of teachers on the mathematics performance and attitude toward mathematics of students, found that the MPK of teachers appears to influence both the mathematics performance and attitude of students. MPK is said to be how teachers relate their subject matter (what they know about what they teach) to their pedagogical knowledge (what they know about teaching) and how subject matter knowledge is a part of the process of pedagogical reasoning (Cochran et al. 1993 as cited in Turnuklu and Yesildere 2007). Sasing then recommends enhancing the MPK of teachers as this may impact student mathematics performance.

In addition, Davis-Langston (2012) mentioned that it is essential to look into teaching styles since there are still a few pieces of research on their quantitative impact on student achievement. In Langston's research on teaching style, teacher efficacy, and student achievement, there was a statistically low positive correlation among teaching styles contributing to mathematics achievement. Hence, interventions, teaching styles, or teacher efficacy may not be the only factors that can help raise students' achievement in mathematics.

In order to investigate the relationship between teaching style and the students' achievement in mathematics, this study specifically sought to answer the following questions:

1. What are the different teaching style profile of teachers?
2. What is the status of students' achievement in mathematics?
3. Is there an association between teaching style and students' achievement in mathematics?

Literature Review

The concept of "teaching styles" has been defined in various ways. Some refer to teaching styles based on the teacher's methods for classroom instruction. For instance, Peterson (1979) describes teaching styles in terms of how teachers use space in the classroom, their instructional activities and materials, and their method of grouping students. Similarly, McNeil and Wiles (1990) express teaching style as various ways in which teachers teach, and according to them, style is reflected in the teacher's attention and methods, such as verbal interaction, questioning, lecture organization, and the like. However, some studies refer to teaching styles based on teachers' behavior as they provide classroom instruction.

For example, Conti (1989) defines teaching style as the overall traits and qualities that a teacher displays in the classroom that are consistent for various situations.

On the other hand, Ornstein (1995) views teaching style as a broad dimension or personality type that encompasses the teacher's stance, the pattern of behavior, mode of performance, and attitude toward self and others. Ellis (1979), on the one hand, views teaching style as the set of models from which various teacher behaviors can be selected based on desired outcomes. Overall, teaching styles vary from teacher to teacher since they are heavily influenced by the teacher's personal qualities, philosophy in life, educational philosophy, and attitude (Beyond Crossroads 2006, as cited by Davis-Langston 2012).

Teaching Style Models

According to the literature, various teaching styles (Alias and Zakaria 2008, Davis-Langston 2012, Ornstein 1995, Paz 2009) are typically based on the critical roles and characteristics of the teachers inside and beyond their classrooms. Although teaching style, by definition, can refer to both methods and behaviors that teachers demonstrate inside the classroom, some studies devised several roles to describe further the teacher's interaction with the students and the teacher's view regarding the content. Some of these teaching style models are proposed by educators and writers as follows:

The model by Thelen et al. (1954) describes teaching styles similar to any one of the following scenarios: *Socratic,* where the teacher's image is a wise and thought-provoking teacher who argues with a student in purpose over a subject matter through artful questioning; *town-meeting,* where the teacher moderates the discussion until students arrive to answers by themselves; *apprenticeship,* where teachers serve as a role model in learning, decision making and life in general; *boss-employee,* teachers who use this style of teaching uses his/her authority and rewards or punishes the students for making sure that work is done; *good-old team,* a teacher in this style uses teamwork among students and plays as a coach in a team.

Meanwhile, Reissman and Silvert (1967) view teaching styles as based on the following: *Compulsive* is the teacher who teaches things over and over and is concerned with functional order and structure; *boomer* is the teacher who has a strong personality and motivates every student with a loud voice; *quiet*, the calm teacher that commands respect and attention; a *coach* is the informal athletic teacher that is physically expressive in the class; *maverick* is the teacher that raises difficult questions and presents disturbing ideas; *entertainer* the teacher who always has a joke and enjoys a laugh with the students; *secular* is the teacher who is relaxed and informal with students; and the *academic teacher* who always shares knowledge and substance of ideas.

Rubin (1971) on the other hand, enumerates the different styles of the teachers as any of the following: *explanatory,* where the teacher is in command of the subject matter; *informative,* where the teacher presents information and students are expected to listen and follow instructions; *interactive,* where the teacher facilitates the development of ideas; *programmatic* where the teacher guides student activities and facilitates independent learning; *inspiratory* where the teacher exhibits emotional involvement in teaching; and *corrective* where the teacher provides feedback by analyzing work, diagnosing for errors, and presenting corrective advice.

Meanwhile, Mosston and Ashworth (1985) argued that teachers could lead students to learn from basic to advanced information to creative problem-solving if they follow a universal and deliberate theory of teaching called the Spectrum of Teaching Styles (Mosston 1966) which investigated its application for some years. They have defined ten teaching styles that are categorized into two structures. The first half is called styles A to E. *Style A(command),* where the teacher decides, and students follow. *Style B (practice),* where students learn the initial decision-making steps because the teacher shifts specific decisions to them. *Style C(reciprocal),* where cooperative learning is practiced, and more decision-making is done by the students as they are given criteria to engage in observing, listening, comparing, contrasting, concluding, and communicating results to a fellow student. *Style D(self-check)* is when learners practice tasks designated by the teacher and evaluate themselves against established criteria. Finally, *Style E (inclusion)* is when each learner chooses options within a given task. The structure of styles A to E created conditions where students learn the basic skills about a subject matter. These involve engaging learners in cognitive skills such as memory recall, identifying, sorting, comparing, contrasting, and the like. Beyond that, the authors call the discovery threshold – an invisible line that divides the ten styles into two structures. The second structure contains styles that help students think more or beyond what is expected. These styles evoke the process of discovery and creativity, styles F-J: *Style F (guided discovery),* where teachers guide learners to unlocking concepts and principles; *Style G (divergent thinking),* where students discover alternatives through teachers' questions, problems or situations; *Style H(individual program learners design)* where a general subject matter is designated by the teacher and lets students discover the designs, questions and problems about it then seek solutions and verify them; *Style I (learner initiated)* this is when students can experience the skills learned from Style A to H, and they begin to initiate learning by making their own decisions but still in communication with their teacher to gain information; *Style J (self-teaching)* a style that happens outside the school where there is no need for a teacher because the learner makes all decisions.

Teaching Style by Grasha

Another established teaching styles model by Grasha (2002) looks at teaching style from the perspective of the following teaching approaches:

- Expert–The expert teacher possesses knowledge and expertise that students need, strives to maintain status as an expert among students by displaying detailed knowledge and challenging students to enhance their competence, and is concerned with transmitting information and ensuring that students are well prepared. The advantage of this style includes the teacher's ability to display information, knowledge, and skills; however, if overused, this can be intimidating to inexperienced students. In addition, the teacher may not always show the underlying thought processes that produce answers.

- Formal Authority–The teacher is concerned with the correct, acceptable, and standard ways to do things, provides positive and negative feedback, and establishes students' goals, expectations, and rules of conduct inside the classroom. The advantage of this style includes students' focus on clear expectations and acceptable ways of doing things. On the other hand, substantial investment in this style can lead to rigid, standardized ways of managing students and their concerns.
- Personal Model–The teacher oversees, guides, and directs by showing how to do things and encouraging students to observe and emulate the teacher's methods. The advantage of this style is that it is "hands-on," and it emphasizes direct observation of the teacher, and thus, the teacher becomes a role model. However, teachers who use this style may believe that their approach is the best way, and students may feel inadequate if they cannot live up to the expectations and standards set by the teacher.
- Facilitator–The teacher guides students by asking questions, exploring options, suggesting alternatives, and encouraging students to develop criteria to make informed choices. The overall goal is to develop independence and responsibility among students. This approach emphasizes the personal nature of teacher-student interactions. Its flexibility, focus on students' needs and goals, and willingness to explore options and alternative courses of action were the mentioned advantages of the facilitator style. However, this style is time-consuming and can be ineffective when a more direct approach is needed. The student's comfort will also be disadvantaged if the style is not done positively.
- Delegator–The teacher is concerned with developing students' capacity to function autonomously. This style contributes to students perceiving themselves as independent learners, but teachers may misread students' readiness for independent work, which is why some students may become anxious when given autonomy.

Grasha's teaching Style Clusters

Grasha's study established that all teachers possess each of the five styles to different degrees. Accordingly, he devised four clusters of the aforementioned styles in his continuous analysis of observations, interviews, and workshop experiences. Table 1 presents the clusters of teaching styles and the recommended activities.

Table 1. *Teaching Styles and Corresponding Activities*

	Cluster 1	Cluster 2	Cluster 3	Cluster 4
Primary Styles	Expert/formal authority	Expert/personal model/formal authority	Expert/facilitator/personal model	Expert/facilitator/delegator
Secondary Styles	Personal model/facilitator/ delegator	Facilitator/delegator	Formal authority/delegator	Formal authority/personal model
Activities	Lectures Term papers Guest presentations AVP Guest speakers Teacher–centered class discussions Strict standards/ requirements Grades/tests emphasized	Demonstrating ways of thinking/doing things Coaching/guiding students Illustrating alternatives Sharing personal viewpoints Sharing thought processes involved in obtaining answers Using personal examples to illustrate content points Having students emulate the teacher's example	Small group discussion Laboratory projects Instructor-designed group projects Student teacher of the day Self-discovery activities Learning pairs/debates Case studies Role plays/simulations Problem-based learning Practicum/guided readings	Student-designed group projects Independent study Independent research projects Position papers Student journals Modular instruction Self-discovery learning projects Cooperative learning activities

Note: Adapted from Grasha (2002, p. 158).

The combination of the high delegator and high expert teaching styles effectively develops conceptual problem-solving skills in chemistry (Paz 2009). Additionally, combining the high delegator and high facilitator teaching styles helps students achieve remarkable scores in conceptual problem-solving in chemistry (Paz 2009). Note that the combinations mentioned are in cluster 4 in Grasha's model.

Davis-Langston (2012) also conducted a related study to explore the relationships among teaching styles, teachers' perceptions of their self-efficacy, and students' mathematics achievement. Again, it was concluded that elementary school teachers' teaching styles and student mathematics achievement had a significant relationship.

The relationship showed that as teachers' perceptions of their use of these teaching styles increased, so did their students' achievement in numbers, operations, and mathematics. Moreover, Davis-Langston (2012) established that the statistically significant correlations found highlight the relationship between the teaching styles (personal model, facilitator, and delegator) and mathematics achievement; thus, clusters 2, 3, and 4 from Table 1. However, a negative correlation was found between the expert teaching style and students' mathematics achievement, which according to the author, harmfully influences student achievement.

Meanwhile, Dunn and Dunn (1978) argued that teaching style could be modified if the instructor understood how to respond to varied learning styles. For example, Hatfield et al. (1997) suggested that student-centered teaching styles be employed in teaching mathematics. It was further mentioned that manipulative materials, explorations, and discovery approaches would work well in teaching mathematics. As seen from Table 1, clusters 3 and 4 are such strategies mentioned. Concerning this, Liwag (2008) recommended that students' prevailing poor mathematics reasoning abilities should be the focus of classroom instruction, and the skills of making generalizations should be developed among students by augmenting exposure to this skill until mastery is done. Researchers should consider the causes of poor mathematics reasoning abilities through observation of the different teaching strategies in a classroom setting. Hence, teachers are encouraged to do curricular innovation that integrates the development of reasoning abilities through experiments.

Incidentally, Guloya (2007) mentioned that mathematics is one of the subjects least liked by students and that teachers must be well-trained in content and strategies. Liwag (2008) stressed that how teachers might develop their students' mathematics reasoning are the subject of many studies, such as the Third International Mathematics and Science Study (TIMSS). However, there has been little cooperative work on how math teachers structure lessons to develop reasoning in mathematics.

Students' Achievement in Mathematics

Based on their research, achievement in mathematics of American students was among the bottom 25 percent of all countries, reflecting a large extent of low teacher coverage of the subject matter. In our country, most research is usually based on interventions. Several studies have focused on developing students' mathematics reasoning, yet the status of mathematics achievement of Filipino students remains very low. Velasco (2013) even contended that for more than a decade, the mathematics thinking of Filipino students has not improved.

In the same way, in research about students' level of mathematical thinking concerning teachers' transformational practices in mathematics courses, Caluya (2000) found that even in relatively high-performing schools, both teachers and students find difficulty in proving theorems as well as solving problems that require complex solutions. Although progress was shown in the student's level of thinking from the original level they posted, only a few students displayed high-level skills. In addition, the progress was influenced by certain traits and practices of teachers like constant exposure to manipulative tasks, problem-solving activities and proving, prompt feedback of assessments, persuasiveness, time management, student-talk, teacher-facilitator scheme, and cooperative learning (Caluya 2000).

In similar research done by Escaran (2005) on investigating hypothetico-deductive reasoning using an inductive approach, although students had been exposed to the preliminaries of proof writing, they still lacked great readiness to deal with the deductive level of formal proof writing in mathematics. Hence, more time was needed for them to develop their proof-writing skills.

On the other hand, Guloya (2007), in her study on software-assisted instruction, students' thinking levels, and achievement in mathematics, found that the students' gain scores in the mathematics achievement test and levels of thinking were not statistically significant. Consequently, using software-assisted instruction does not necessarily improve students' achievement. Liwag (2008) likewise examined the mathematics thinking and reasoning abilities of 319 and 207 students from private and science high schools in Quezon City. It was found that the overall performance of students from both schools was at 50%, suggesting low thinking and reasoning abilities in mathematics. Meanwhile, Buendicho (2009) studied the effects of student-initiated questions on reasoning ability and mathematics achievement. It was reported that there was no significant between the student-initiated-questioning group and the teacher-posed-questioning group's mathematics achievement. Moreover, Solaiman et al. (2017) confirmed the findings of many researchers (Genz 2006, Tan and Yebron 2008, Dindyal and Besoondyal 2007) and found that not one of the 409 respondents reached the expected level of mathematics understanding of a student before entering the third year high school.

Furthermore, Lappan (1999) studied the data from TIMSS and the National Assessment of Education Progress (NAEP), showing that student performance in mathematics at all levels is quite alarming. Mistretta (2000) even established how students are not demonstrating strong conceptual knowledge of the subject. Incidentally, both old and current researches consistently show Filipino students' low performance in mathematics. Guloya (2007) also argued that mathematics is one of the least-liked subjects by students.

Methodology

This study investigated the relationship between teachers' teaching styles and student achievement. These objectives were achieved through a non-experimental quantitative method that utilized the correlational research design. Moreover, the study analyzed data gathered through researcher-made and adapted tests. The data gathered from the survey were used to find out (1) the teachers' teaching style and (2) the student's achievement.

In selecting the participants, multistage cluster sampling was performed. In this study, the stages involved selecting districts in the Division, followed by selection of schools, teachers, and students. The two districts in Division has twenty-one public secondary high schools, of which 15 are in Rodriguez and six are in San Mateo. From these 21 schools, only 13 schools were initially selected as these were accessible via public transportation. However, three schools declined, and thus these were excluded from the list. Therefore, 10 or 48% of the 21 schools comprised the sample. As can be seen in Table 2, seven schools (R1 to R7) were from Rodriguez District, and three schools (S1 to S3) were from San Mateo District. After selecting the schools, all the grade 9 mathematics teachers on the list were personally invited to participate in the study. The teachers were informed about the researcher's institutional affiliation and contact information, the purpose of the research, the confidentiality involved, and the instruments to be administered. The participation of the respondents in the study was voluntary. Teacher participants were also informed that they could decline at any time if they wished to. They were also informed that one of their grade 9 classes would be part of the study.

Table 2. *Demographic Profile of Participants*

School	No. of Teachers	Teachers' Gender		Teaching Experience (in years)				No. of Students
		Male	Female	1–5	6–10	11–15	16–20	
R1	1	0	1			1		51
R2	3	2	1		3			130
R3	3	1	2		3			151
R4	3	2	1	1			2	164
R5	3	1	2		3			167
R6	2	0	2			1	1	99
R7	6	2	4	4	1	1		293
S1	2	0	2		2			105
S2	3	1	2		3			136
S3	4	1	3		1	2	1	190
Total	30	10	20	5	16	5	4	1489

As shown in Table 2, a total of 30 mathematics teachers and one of their grade 9 classes comprised the participants of the study. Of these teachers, 21, or 70%, were from Rodriguez district while 9, or 30%, were from San Mateo; 20, or 66.7%, were female while 10, or 33.3%, were male, and 5, or 16.7% have been teaching for less than five years while 25 or 83.3% have been in the profession for more than five years. In all, a total of 1489 students participated in the study. They belonged to the respective grade 9 classes of the 30 teacher-participants. Each class had an average size of 49 students.

Instruments

Teaching Style Inventory

The Teaching Style Inventory Version 3.0 by Grasha (1996) was adapted for use in identifying teachers' teaching styles in this study. It is an open-access instrument tagged by Dr. Carrie Myers, the program leader for the College Teaching Certificate of the Graduate School of Montana State University in Montana, the US, where Grasha was once affiliated. The instrument is a 40-item test that employed a Likert-style response format, with options ranging from 1 (strongly disagree), 2 (disagree), 3 (somewhat disagree), 4 (neither agree nor disagree), 5 (somewhat agree), 6 (agree), and 7 (strongly agree) for responses. The questions are designed to categorize the various teaching styles, namely, (a) expert, (b) formal authority, (c) personal model, (d) facilitator, and (e) delegator.

The test was revised to suit the styles and situations of a mathematics teacher. Three experts, one in the field of psychology, one in mathematics, and one in educational research, validated the content and structure of the instrument. Revisions were made based on the comments and suggestions of the experts until the instrument was ready for pilot testing. After the pilot testing of the revised instrument, the scale reliability was measured using Cronbach's alpha to test the internal consistency of the items. Results showed a reliability coefficient of 0.803, suggesting a high internal consistency.

Responses were then scored according to Grasha's scoring key. Finally, mean scores were computed according to the dimensions (expert, formal authority, personal model, facilitator, and delegator) as shown in Table 3.

Table 3. *Teaching Styles*

Teaching Style/Norm	Low	Moderate	High
Expert	1.0–3.2	3.3–4.8	4.9–7.0
Formal Authority	1.0–4.0	4.1–5.4	5.5–7.0
Personal Model	1.0–4.3	4.4–5.7	5.8–7.0
Facilitator	1.0–3.7	3.8–5.3	5.4–7.0
Delegator	1.0–2.6	2.7–4.2	4.3–7.0

Note: Adapted from Grasha (1996).

Achievement Test

An achievement test was created based on Grade 9 third-quarter competencies in the mathematics curriculum. The topics included in the test were quadrilaterals, triangle similarity, and Pythagorean Theorem. The achievement test was composed of 35 multiple-choice items validated by a panel of content experts in mathematics. The test was pilot tested to 30 students. Using the KR-20 test, a reliability coefficient of 0.77 was obtained, clearly suggesting an acceptable and good internal consistency of the test items.

Results and Discussion

The Teachers' Teaching Styles

Discussion of the results related to teaching style profiles of the teacher-participants is based on the (a) mean score of teaching styles; (b) degree to which they manifest each teaching style; (c) teaching style with the highest mean score; and (d) frequency of teacher-participants per cluster of teaching style.

Table 4. *Teaching Style based on Mean*

Teaching Styles	$\bar{x}$	SD	Descriptive Value	Norm (High)
Expert	5.99	0.38	High	4.9-7.0
Formal Authority	5.85	0.42	High	5.5-7.0
Personal Model	5.77	0.41	Moderate	5.8-7.0
Facilitator	5.63	0.53	High	5.4-7.0
Delegator	5.78	0.44	High	4.3-7.0

Table 4 shows the mean score of all teachers in a particular teaching style. Descriptive information shows that the highest mean and descriptive value is "expert" at 5.99, with a descriptive value of high. Additionally, "formal authority" has a mean score of 5.85 (high), "delegator" has a mean score of 5.78 (high), and "facilitator" at 5.63 (high). On the other hand, only the "personal model," whose mean is 5.77, has a descriptive value of moderate.

The "expert" style represents a teacher who transmits knowledge to students (Grasha 2002). In this style, all information regarding the subject would come from the teacher. Canto-Herrera and Salazar-Carballo (2010), in their research about the relationship of teachers' beliefs and teaching style to student achievement in mathematics, found the same result, where the "expert" teaching style had the highest mean. The same result was found by Magulod Jr. (2017) in Cagayan among pre-service mathematics teachers. All these results confirm Grasha's claim that teachers in mathematics usually apply the "expert" teaching style.

It is also good to note that "expert" and "formal authority" teaching styles are popular in situations where the class size is large, there is a timetable for covering a topic or material, and teachers have to prepare their students for standardized exams (Grasha 2002). As observed during the data gathering, these are the usual teaching scenarios in the Philippine education system: large class size, with some class sizes ranging from 40-60 students, 49 on average in this study, and time table covering the topic. Moreover, the administration of the achievement test was moved toward the latter part of the study as requested by the teachers because, according to them, the lessons covered by the test were not yet discussed in some classes. Therefore, the styles "expert" and "formal authority" were among the teaching styles with the highest mean because it is the most convenient in this situation. However, when these styles are followed, knowledge retention may be short (Grasha 2002).

On the other hand, Grasha's (2002) conceptual model of teaching style further established that all teachers possessed each of the five styles in varying degrees. This study looked into the different degrees to which teachers manifest the five styles (see Table 5). Based on the results, it can be inferred that in terms of teaching style degree, a majority (70% or more) of the teachers scored high in all categories, except for "personal model," where there are only 13 or 43.3% who scored high and 17 or 56% who scored moderately. Moreover, the 13 teachers who scored high in the "personal model" gave the highest mean rating (6.14) among all categories.

Table 5. *Teaching Style Based on Degree*

Categories	Degree	*f*	%	$\bar{x}$	SD
Expert					
	High	30	100	5.99	0.38
Formal Authority					
	High	25	83.3	5.98	0.35
	Moderate	5	26.7	5.25	0.13
Personal Model					
	High	13	43.3	6.14	0.23
	Moderate	17	56.7	5.48	0.23
Facilitator					
	High	21	70	5.88	0.34
	Moderate	9	30	5.04	0.41
Delegator					
	High	30	100	5.78	0.44

Accordingly, from Table 5, the "personal model" is the style with the lowest descriptive value (moderate). "Personal model" is said to be the teaching style where the teacher becomes a person to emulate by the students, and according to Grasha (2002), this style is often used in theatre arts. A contradicting result was found by Atasoy et al. (2018) about the relationship between mathematics teachers' instructional styles and educational philosophical background, where the "personal model" was perceived to be the highest teaching style preferred by the participants. These researchers from Turkey believed that the confidence of teachers in this style (personal model) is complete.

In addition, the teachers' teaching style profile in this study was determined based on the highest mean score in the five teaching style categories. Results (see Table 6) indicated that while the majority (76.7%) of the teachers manifest a single teaching style, some (23.3%) teachers scored highest in more than one or a combination of teaching styles. It is also interesting to mention that the results (based on Tables 5 and 6) confirm Grasha's conceptual model of teaching style that a teacher may possess more than one teaching style in varying degrees.

Table 6. *Teaching Styles Based on Highest Mean Score*

Teaching Styles	***f***	**%**
Expert	12	40.0
Formal Authority	5	16.7
Personal Model	1	3.3
Facilitator	1	3.3
Delegator	4	13.3
Total	23	76.7
Expert, Delegator	2	6.7
Expert, Formal Authority	1	3.3
Expert, Formal Authority, Personal Model	3	10.0
Expert, Facilitator, Delegator	1	3.3
Total	7	23.3
Grand Total	30	100.0

Recall that Grasha's continuous analysis of teachers' observations, interviews, and workshop experiences gave rise to four clusters of the aforementioned teaching styles. Thus, the teachers in this study were further classified according to the same clusters Grasha (2002) described based on their primary teaching style or teaching style/s rated with the highest degree. However, as shown in Table 7, results indicate that while some teachers have been classified according to the clusters defined by Grasha (2002), the others were not assigned to any of the four clusters.

Table 7. *Teaching Styles based on Clusters*

Cluster and Teaching Style	*f*	%
Cluster 1 Primary Style: Expert and Formal Authority	4	13.4
Cluster 2 Primary Style: Expert, Personal Model, Formal Authority	7	23.3
Cluster 3 Primary Style: Expert, Facilitator, Personal	0	0
Cluster 4 Primary Style: Expert, Facilitator, Delegator	9	30.0
Cluster 5 Primary Style: Formal Authority, Personal Model, Delegator	1	3.3
Cluster 6 Primary Style: Expert, Formal Authority, Delegator	3	10.0
Cluster 7 Primary Style: Expert, Delegator	6	20.0
Total	30	100.0

There are 4, or 13.4%, teachers who fall under cluster 1 (expert, formal authority); 7, or 23.3%, fall under cluster 2 (expert, personal model, formal authority); none, or 0%, fall under cluster 3 (expert, facilitator, personal model), and 9 or 30% fall under cluster 4 (expert, facilitator, delegator). Two-thirds of the teachers, 20 or 66.67%, fall under the four clusters; surprisingly, 10, or 33.33%, are not in any of the clusters. It means that their primary teaching styles are not any of the 4 clusters defined by Grasha. This result contradicts Grasha's (2002) study, where 92% of them fit in the four clusters. Hence, this study tried to determine their primary styles and came up with clusters 5, 6, and 7. From Table 7, 1 or 3.3% fall under cluster 5 (formal authority, personal model, delegator), 3 or 10% fall under cluster 6 (expert, formal authority, delegator), and 6 or 20% fall under cluster 7 (expert, delegator).

From Grasha's interpretation, clusters 5 to7 are a combination of student-centered and teacher-centered styles. Incidentally, some studies using Grasha's model have found an effective combination of teaching styles, although they were not in any of the four clusters. For instance, in local research about teaching styles and student performance in conceptual and algorithmic problem-solving in chemistry, Paz (2009) found that the combination of the high delegator and high expert teaching styles is effective in the development of conceptual problem-solving skills and helps in achieving a remarkable score in both conceptual and algorithmic problem-solving in chemistry. On the other hand, teachers with expert and delegator teaching styles do not fall in any of Grasha's clusters (see Table 7).

In addition, Davis-Langston (2012), in his research about the relationship among teaching style, teachers' perceptions and teachers' self-efficacy, and students' mathematics achievement, found a significant positive correlation in two other clusters (personal model, facilitator, delegator; formal authority, facilitator) not in any of the clusters in Grasha's model. Note that the four clusters of styles were obtained from a systematic study based on analysis of classroom observations and interviews with teachers and the responses of several hundred workshop participants who related the five styles to the instructional processes they employed in the classroom (Grasha 2002). However, in this study, the new clusters may also give rise to a combination of teaching styles that may help student achievement in the Philippine setting. Moreover, the researcher listed possible activities a teacher may do in the new clusters discovered in this study using Grasha's teaching styles and related activities on Table 8.

From another angle, it can be observed based on the suggested activities that clusters 3 and 4 suggest student-centered teaching styles. In addition, Hatfield (1997) argued that student-centered teaching styles like those in either clusters 3 and 4 work well in teaching mathematics. In addition, the suggested activities by Grasha (2002) answer the stipulated learning principles and theories in the framework of the mathematics education system by Department of Education (2016). These learning principles, discovery and inquiry-based learning, experiential and situated learning, reflective learning, constructivist learning, and cooperative learning, guide mathematics teachers and enable their learners to achieve mathematics education's twin goals: critical thinking and problem-solving.

Table 8. *New Clusters of Teaching Styles and Corresponding Activities*

	Cluster 5	**Cluster 6**	**Cluster 7**
Primary Styles	Formal Authority, Personal Model, Delegator	Expert, Formal Authority, Delegator	Expert, Delegator
Activities	Demonstrating ways of thinking/doing things Independent study Using personal examples to illustrate content points Having students emulate the teacher's example Teacher–centered class discussions Strict standards/requirements Grades/tests emphasized	Lectures Teacher–centered class discussions Strict standards/requirements Grades/tests emphasized Self-discovery learning projects Cooperative learning activities	Lectures Term papers Teacher–centered class discussions Small group discussion Laboratory projects Student teacher of the day Self-discovery activities Case studies Role plays/simulations

Note: Adapted from Grasha (2002, p. 158).

Student Achievement

The achievement test was administered to the student participants from one of the intact classes handled by each of the teacher participants. The classification of mastery level is based on the quartile distribution of mean percentage scores (MPS) among schools by Department of Education (2019), in which the national standard level of acceptable MPS for all subject areas is 75%.

Table 9. *Students' Mastery Level in Achievement Test*

	Per Student		**Per Class**	
Level	*f*	%	*f*	%
Superior (76–100%)	9	0.60	0	0
Average Mastery (51–75%)	242	16.26	2	6.67
Low Mastery (26–50%)	678	45.53	26	86.66
Poor (0–25%)	560	37.61	2	6.67
Total	1489	100.0	30	100.0

Note: Students' raw scores and class mean percentage scores are used to determine the frequency in each quartile.

As can be seen in Table 9, based on individual raw scores, results indicate that only nine students or 0.6% can reach the superior level of mastery. 242, or 16.26%, are at the average level, 678, or 45.53%, are at a low level, and 560, or 37.61%, are at a poor level. Most 1238 or 83.14% student participants fall below the 75%-acceptable level. Accordingly, based on class mean percentage scores (MPS), none of the intact classes are at a superior level. Only 2, or 6.67%, fall under the average level, 26, or 86.66%, under the low level, and 2, or 6.67%, under the poor level of mastery. Remarkably, not even 1% of the intact classes can reach the superior level of mastery. When grouped per class, 28, or 93.33%, of the classes are below the acceptable level, while only 2, or 6.67%, are at the average mastery level.

The results shown in Table 9 confirm the reports of several studies (Buendicho 2009, Caluya 2000, Escaran 2005, Guloya 2007, Liwag 2008, Solaiman 2013, Velasco 2013), which emphasized that only a few numbers of students achieved the mastery level in mathematics. This result has been the scenario for decades and needs so much attention.

In particular, the importance of teaching and learning mathematics was established at the beginning of this paper, especially since it plays a vital role in the field of mathematics and life in general. Several local studies mentioned similar thoughts on the importance of developing students' mathematical skills because it contributes to their critical thinking and problem-solving (Buendicho 2009, Liwag 2008, Solaiman et al. 2017). Upon confirming results on students' achievement in the last decades to be below mastery level, it is suggested by this study to look into some teacher factors related to students' achievement in mathematics since it was established that teachers' mathematics competencies are critical to the effective teaching of the subject (Ndlovu 2014).

Teachers' Teaching Style and Students' Achievement in Mathematics

This section examines the relationship between teachers' teaching styles based on Grasha's model and students' achievement in mathematics. Two teaching style profiles were used to analyze the data: (1) based on clusters, as described in Table 7, and (2) based on the highest mean, as described in Table 6. Prior to the test of association, a one-way analysis of variance was used to examine whether the students' achievement is a function of teaching style.

For the first set of analyses in this section, the researcher utilized the data on the respondents' primary teaching style or teaching style/s rated with the highest degree to determine the teachers' corresponding cluster. The categories of teaching style were: (1) cluster 1 (expert, formal authority); (2) cluster 2 (expert, personal model, formal authority); (3) cluster 4 (expert, facilitator, and delegator); and (4) cluster 0 (those who did not fall in any of the four clusters). Still, the dependent variable was the students' MPS on the researcher-made achievement test.

It was made sure that the assumptions for independence, normality, and homogeneity of variance were met before the one-way ANOVA was generated. The results of the one-way ANOVA indicate no significant difference (3, 26)= 1.120, $p<0.359$ at a 0.05 level of significance (see Table 10). This result shows no significant differences in students' achievement test scores among the clusters of teaching styles. This result may mean that student achievement is the same regardless of teachers' teaching styles.

Table 10. *ANOVA of Teaching Styles and Students' Achievement in Mathematics*

	Sum of Squares	*df*	Mean Square	*F*	*p*
Between Groups	43.62	3	14.54	1.120	0.359
Within Groups	337.30	26	12.97		
Total	380.92	29			

*p<0.05.

That being the case, this result failed to accept the research hypothesis that there is a significant relationship between teachers' teaching styles and students' mathematics achievement. Furthermore, this result contradicts that of Paz (2009), that student-centered teaching styles (clusters 3 and 4) help students achieve a remarkable score in conceptual problem-solving. In addition, the result is also contrary to what Langston (2012) concluded in his study, that positive correlations exist among the three teaching styles found in clusters 3 and 4 of Grasha's conceptual model of teaching style and student achievement.

Still, Goodman and Kruskal's gamma was run to determine the association between teaching style based on the clusters and students' achievement (see Table 11). Results indicate no correlation between the two variables (G=0.083, p<0.622). This result means that teacher's teaching style is not associated with student achievement in mathematics, leading to the rejection of the research hypothesis.

Table 11. *Correlation of Teaching Style and Students' Achievement*

	Value	Asymp. Std. Error	Approx. T	*p*
Gamma	0.083	0.167	0.494	0.622
N of Valid Cases	30			

*p<0.05.

The results from the previous section prompted the researcher to do another analysis to test the association between teachers' teaching style and students' achievement using the teaching style profile based on the highest mean score (see Table 6). This time, the teachers were grouped according to the teaching style where they scored the highest, and if the teachers scored highest in more than one teaching style, they were classified under the combination category. For this profile, the categories of teaching style were (1) expert, (2) formal authority, (3) personal model, (4) facilitator, (5) delegator, and (6) combination. The dependent variable was the students' MPS on the researcher-made achievement test, the same as the data used in the first section.

Although assumptions for independence and normality were met, the result of Levene's test for homogeneity of variance shows a significant difference between the variances of the teaching style and student achievement at F (3, 24)=3.77, p>0.024. This result means that the assumption for homogeneity of variance was not met. Newsom (2020) explained that when the assumption for homogeneity of variance is not met, and there are unequal group sizes, Welch's test is an alternative test for ANOVA. Newsom (2020) cited Welch (1951) that it is good to consider Welch's test in cases when sample size and variance are not equal. Hence, for this particular data, Welch's robust test results indicate a significant effect, Welch's F (3, 9.39)=7.071, p<0.009 at a 0.05 level of significance. This result shows significant differences in students' achievement test scores among the teaching styles according to the highest mean. This result means that student achievement is a function of teachers' teaching styles. The estimated omega squared (ω^2=0.39) indicates that approximately 39% of the total variance in students' achievement is attributable to the difference between the teaching styles based on the highest mean scores.

Table 12. *Post Hoc Results for Students' Achievement by Teaching Styles Based on Highest Mean Score*

		Mean Differences ($\bar{X}_1$ - $\bar{X}_2$) (*p* value and effect sizes are indicated in parentheses)			
Teaching Style	Mean	Expert	Formal Authority	Delegator	Combination
Expert	41.96				
Formal Authority	29.55	12.41 (*p* = .011)* (1.59)			9.80 (*p* = .036)* (1.86)
Delegator	45.16				
Combination	39.55				

*p<0.05.

Post hoc comparisons using Games-Howell test procedures were likewise used to determine which pairs of the four group means differ significantly. The results shown in Table 12 indicate that students whose teachers have a high mean score on expert style (M=41.96, SD=10.71) have a significantly higher achievement score than students whose teachers have a high mean score on formal authority (M=29.55, SD=2.80) style of teaching. The effect size is 1.59, and the result is statistically significant at p=0.011. Additionally, those students whose teachers have a high mean score on a combination of teaching styles (M=39.35, SD=6.9) have significantly high average MPS scores than those students whose teachers have a high mean score on formal authority (M=29.55, SD=2.80) style of teaching. This result is also statistically significant at p = 0.036 with an effect size of 1.86.

However, when Goodman and Kruskal's gamma was run to determine the association between teaching style (highest observed mean) and students' achievement (MPS), it was found that there is no correlation between the two variables (G=-0.047, p<0.778) (see Table 13).

Table 13. *Correlation of Teaching Styles Based on Highest Mean Score and Students' Achievement*

	Value	**Asymp. Std. Error**	**Approx. T**	***p***
Gamma	-0.047	0.169	-0.282	0.778
N of Valid Cases	28			

**p<0 .05.*

Not statistically significant results suggest looking into other factors such as the teachers' awareness of how students learn best, continuous training, developing the reasoning ability of students, and even looking into the affective factors as it can lead to new reasons that may explain the poor mathematics reasoning abilities of Filipino students (Guloya 2007, Liwag 2008, Solaiman et al. 2017, Ndlovu 2014). Moreover, these results confirm that other teacher factors may be related to student achievement (Mahmud and Salimian 2012, as cited in Kušen and Marinović 2013).

Grasha (2002) did address not only teaching styles but also studied the different learning styles of students. According to him, looking into the different learning styles does not mean that what teachers do in the classroom is not significant; however, there is a need to integrate both teaching and learning styles to see success in student achievement further. Thus, teachers may also consider students' different learning styles to apply the most suitable teaching styles for students.

Conclusions and Recommendations

This study has found that although teachers have high scores in effective teaching styles such as expert, facilitator, and delegator, student mathematics achievement remains below the mastery level. It is noted that not even one percent can reach the superior level. Additionally, although there is no significant relationship between teachers' teaching styles when grouped according to clusters and students' achievement in mathematics, a significant relationship was found between teachers' teaching styles when grouped according to the highest mean score and student achievement in mathematics.

In this regard, other teaching styles, especially in the post-pandemic period aside from Grasha's model, may be investigated to find other ways to improve student achievement. Apart from this, other factors may be looked into in future research, such as limited time of lectures, class size, learning environment, study habits, and the usual disruption of classes due to valid reasons such as typhoons, calamities, and epidemic disease and viruses to improve student achievement in mathematics.

Future researchers may increase the samples and expand the population to include not only one cluster in the division. Alternatively, another study may focus on other content areas and grade levels as this will give a different viewpoint of teaching styles and help determine which grade level the student achievement started to decline since this study is limited to grade 9 lessons only.

On the part of the administration, the Department of Education may ensure maximizing academic learning time by training pre-service teachers and re-tooling in-service teachers to develop teaching styles that will leverage student learning in mathematics.

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Properties of 3-Triangulations for *p*-Toroid

*By Milica Stojanović**

In this paper, a method for constructing a toroid and its decomposition into convex pieces is considered. A graph of connection for 3-triangulable toroid is introduced in such a way that these pieces are represented by graph nodes. It is shown that connected, nonorientable graph can serve as a graph of connection for some of the toroids. The relationship between graphs that can be realized on surfaces of different genus and corresponding toroids is considered.

Keywords: *3-triangulation of polyhedra, toroids, piecewise convex polyhedra, graph of connection*

Introduction

Polyhedron and *d*-dimensional polytope are generalizations of the term polygon in 3-dimensional and *d*-dimensional space. We can also generalize the process of triangulation into higher dimensions. Originally, triangulation is dividing a polygon with n vertices by $n-3$ diagonals into $n-2$ triangles and this can always be done. We shall use the same term - triangulation for its generalization in higher dimensions, or more specifically 3-triangulation, *d*-triangulation. In these cases, using only the original vertices, for 3-triangulation we divide the polyhedron into tetrahedra and for *d*-triangulation the *d*-polytope into *d*-simplices. Triangulation problems especially in 2- and 3-dimensional space and other types of polyhedron decomposition have significant applications in engineering and other fields of research (Zhang et al. 2018, Zhang et al. 2020).

But even for triangulation in 3-dimensional space, two new problems arise. The first is that it is not possible to triangulate certain non-convex polyhedra. One example is the famous Schönhardt's polyhedron (Schönhardt 1928). Another problem is, although it is possible to triangulate all convex polyhedra, different 3-triangulations of the same can have different numbers of tetrahedra (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005). This is the reason to consider the smallest (minimal) and the largest (maximal) number of tetrahedra in triangulation. It is shown that such values, linearly, resp. squarely depend on the number n of vertices.

Some properties of 3-triangulation for *p*-toroids will be considered, when triangulation is possible. A polyhedron topologically equivalent to a *p*-torus (i.e., sphere with p handles, $p \in N$ is a given natural number) is a *p*-toroid. The inspiration for this consideration was Szilassi (2005) definition of the torus-like polyhedron, which he called toroid. Here term "toroid" will be used as a common name for *p*-toroids for any $p \in N$, and Szilassi's toroid would be called 1-toroid.

Under certain conditions, it is possible to 3-triangulate some toroids, although there are not convex. Examples of such 1-toroids are given in (Bokowski 2005,

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Császár 1949, Szilassi 1986, 2005, 2012); e.g., the Császár's polyhedron is 1-toroid with the smallest number of vertices. It has 7 vertices and it is triangulable with 7 tetrahedra. It was also discussed as a polyhedron without diagonals (Császár 1949, Szabó 1984, 2009). Additional examples of toroids are given in (Stojanović 2015, 2017, 2021, 2022) and some properties of their 3-triangulations are considered.

Here, after a brief overview of the previous results and the definitions of the necessary terms, a method for constructing a toroid based on a given graph as its graph of connection is given. Examples of p-toroids obtained in the introduced way will then be given with a discussion on the number p of the handles. Also for such toroids, the numbers of vertices and tetrahedra necessary for its 3-triangulation are calculated. These examples show that the number of tetrahedra is such that the lower limit given in the Theorem 1 (Stojanović 2022) is tight.

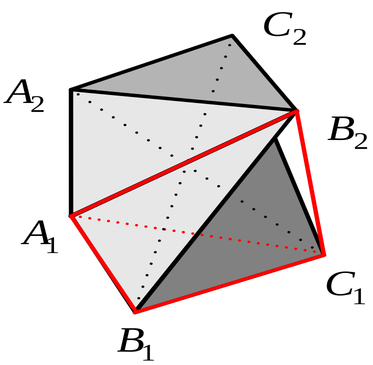
Preliminaries

The general properties of 3-triangulation of polyhedra are given first, and then properties for toroids. Necessary terms are introduced together with previously proven statements. Several examples are given to illustrate the introduced properties.

3-Triangulations of Simple Polyhedra

Although it is possible to triangulate all convex polyhedra, this is not the case with some of non-convex ones. A famous example of such a non-convex polyhedron was given by Schönhardt (1928) and shown in Figure 1. To obtain this polyhedron, we start with the trigonal prism $A_1B_1C_1A_2B_2C_2$ and triangulate its lateral faces with diagonals A_1B_2, B_1C_2 and C_1A_2. After that, we 'twist' the top basis $A_2B_2C_2$ for a small amount in the positive direction. Then none of the tetrahedra with vertices in the set $S = \{A_1, B_1, C_1, A_2, B_2, C_2\}$ would be inner. For example, the tetrahedron $A_1B_1C_1B_2$ has an edge C_1B_2 outside the Schönhardt's polyhedron. Tetrahedra $A_1C_1A_2B_2$ and $B_1C_1B_2C_2$ also contain the edge C_1B_2. For other tetrahedra with vertices in S, the situation is combinatorially the same as in some of the previous cases. Therefore, this polyhedron cannot be triangulated.

Figure 1. *Schönhardt Polyhedron*



Considering the smallest number of tetrahedra in the 3-triangulation of a polyhedron with n vertices we got that it is $n - 3$. E.g. such a polyhedron is pyramid V_{n-1} with $n - 1$ vertices in the basis and the apex, i.e., a total of n vertices. We can 3-triangulate it as follows: do any 2-triangulation of the basis into $(n - 1) - 2 = n - 3$ triangles. The apex together with each of such triangles makes one of the tetrahedra in 3-triangulation.

The triangular prism Π with bases $A_1B_1C_1$ and $A_2B_2C_2$ has 6 vertices and is also 3-triangulable with 3 tetrahedra. Actually, a triangular prism Π can be considered as a 'pyramid' with apex A_2 and spatial pentagon $A_1B_1B_2C_2C_1$ as the basis.

But not all polyhedra have the same property to have 3-triangulation with $n - 3$ tetrahedra. For example, bipyramids with $n - 2$ ($n \geq 5$) vertices in the basis and two apices can be triangulated in two different ways so that triangulations have respectively $2(n - 4)$ and $n - 2$ tetrahedra. Thus, a bipyramid with vertices A, B, C in the basis and apices V_1 and V_2 can be divided into two pyramids $ABCV_1$ and $ABCV_2$ in the first triangulation or into ABV_1V_2, BCV_1V_2 and CAV_1V_2 in the second. A special case of the bipyramid is the octahedron - a polyhedron with 6 vertices. It always gives 4 tetrahedra in 3-triangulation.

3-Triangulations that give small and especially minimal number ($T_{\min}$) of tetrahedra are examined in (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005).

Toroids and 3-Triangulation

We shall start with the term *p-torus*. In surface theory, it is defined as a cyclic polygon with paired sides. Any side s and its pair S are oppositely directed related to the fixed orientation of the polygon and then glued together. By a standard combinatorial procedure - the polygon can be divided and glued to a cyclic normal form $a_1b_1A_1B_1a_2b_2A_2B_2...a_pb_pA_pB_p$, as a p-torus. This combinatorial procedure is independent of the future spatial placement of the surface. So, from any spatial knot (as a topological circle in the space) we can form a p-torus. Of course, its surface can be 2-triangulated to be a surface of polyhedron.

Based on Szilassi's (1986) definition the term p-toroid is introduced (Stojanović 2021, Stojanović 2022).

Definition 1. *A polyhedron solid is called* p-toroid, $p \in N$, *if it is topologically equivalent to a sphere with p handles* (*p-torus*).

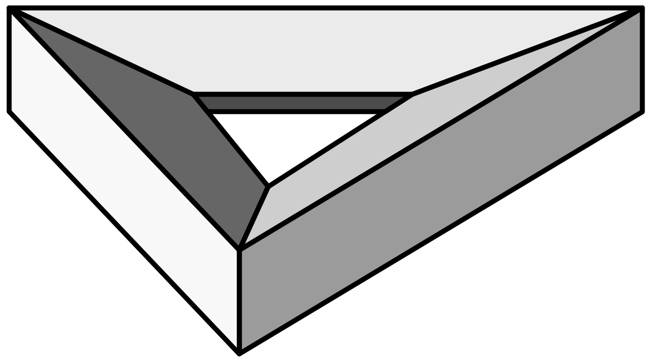
As mentioned earlier, the term *toroid* will be used here as a common name for all p-toroids.

Császár's polyhedron is an example of a 1-toroid with the smallest number of vertices - 7. Its skeleton is the full graph with seven vertices that can be drawn on the torus, and so it has no diagonals. In Wolfram Demonstrations Project Szilassi (Szilassi 2012) shows that Császár's polyhedron is 3-triangulable with 7 tetrahedra and it is a 1-toroid.

In (Szilassi 2005) Szilassi introduced *regular* 1-toroid. It is 1-toroid whose each face has a edges, and exactly b edges meet at each vertex. There are three classes of regular toroids, according to the number of edges incident with each face and each vertices. These classes are: T_1 where a=3, b=6, T_2 where a=4, b=4, T_3 where a=6, b=3.

The Császár's polyhedron is an example of a 1-toroid from class T_1. An example of a regular 1-toroid from class T_2 is given in Figure 2, marked here with P_9, since it has 9 vertices. We can see that the P_9 has 18 edges and 9 faces.

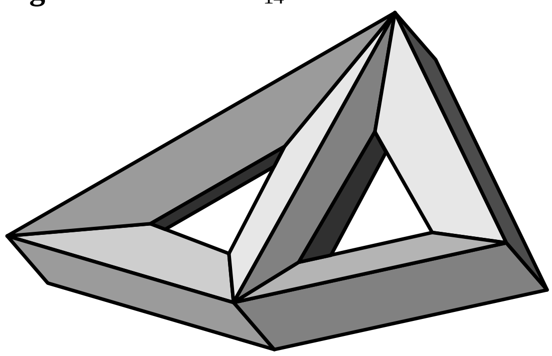
Figure 2. 1-*Toroid* P_9



Source: Stojanović 2015.

An example of 2-toroid P_{14} given in (Stojanović 2017) is shown in Figure 3. It consists of two glued P_9 and has 14 vertices, 32 edges and 16 faces.

Figure 3. 2-*Toroid* P_{14}



Source: Stojanović 2017.

Piecewise Convex Polyhedron and its Graph of Connection

Since toroids are not convex when considering their 3-triangulations, we shall use the following definitions.

Definition 2. *A polyhedron is* piecewise convex *if it can be divided into finitely many convex polyhedra P_i, i = 1, …, m, with disjoint interiors. A pair of polyhedra P_i, P_j is said to be* neighbouring *if they have a common face called* contact face.

If the polyhedra P_i and P_j are not neighbouring, they may have a common edge e or a common vertex v. This is possible iff there is a sequence of neighbouring polyhedra P_i, P_{i+1}, …, $P_{i+k} \equiv P_j$ such that the edge e, or the vertex v belongs to each contact face f_l common to P_l and P_{l+1}, $l \in \{i, \ldots, i + k - 1\}$. Otherwise, the polyhedra P_i and P_j do not have common points.

Remark 1. *Since a convex polyhedron can be* 3*-triangulated, the same holds for a piecewise convex one, especially for a piecewise convex toroid.*
Remark 2. *Each* 3*-triangulable polyhedron is a collection of connected tetrahedra, so it is piecewise convex.*

In our investigation, we shall use the graph of connection for a piecewise convex polyhedron.

Definition 3. *If the polyhedron P is piecewise convex its* graph of connection (*or its* connection graph)*, is a graph with* nodes *representing convex polyhedra P_i, i =* 1*, …, m, pieces of P, and* edges *representing contact faces between them.*

It is important to mention that the division of a polyhedron into convex pieces is not necessarily unique.

In order to have the same number of handles for the considered toroid P and the number of basic cycles of the corresponding connection graph, we shall introduce the term *optimized graph* of connection. Namely, it may happen that in the connection graph made as before, exists some 'false' cycle that do not correspond to some handle of P. Such a situation will disturb us. So, let us consider a toroid P and its graph of connection G that have one or more false cycles. Take all the nodes that belong to the same false cycle of G and the corresponding convex pieces of P. The union of such convex pieces builds a new node of the optimized graph G^*. In such a way we shall make new nodes for all the false cycles. The other nodes of the graph G remain in G^* and we shall call them the old ones. The set of edges for G^* consists of the previous edges between the old nodes, and the edges of G between some old node and some node belonging to a false cycle converted to the edge of G^* between that old node and the new one.

Note that it is not necessary for the new nodes of the optimized graph to correspond to convex polyhedra, they only correspond to simple piecewise convex polyhedra.

In earlier papers of the author (Stojanović 2015, Stojanović 2017, Stojanović 2022) there were proved the theorems for 1-toroids, 2-toroids and p-toroids about the minimal number of tetrahedra necessary for their 3-triangulation. Here we shall mention this one about p-toroids.

Theorem 1. *If a p-toroid with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is* $T_{\min} \geq n + 3(p-1)$.

In Stojanović (2021) it was considered how to construct a toroid P starting from a given graph G, in such a way that G would be a graph of connection for P. It is also shown that if for some simple polyhedron S the graph G is its skeleton, then the number p of the handles of the appropriate toroid P is the same as the number of faces f of S minus one, i.e. $p = f - 1$. Actually, to the question of whether the graph G has to be considered as "planar" or "spherical", answer is that it has to be "planar". Therefore, "outer face" surrounding these kinds of graphs should not be taken into account as some representing a handle.

Constructing *p*-Toroid from a Given Graph of Connection

Further properties and methods of constructing toroids based on the given graph will be considered here.

Theorem 2. *If the graph G is the skeleton of some polyhedron π (not necessarily simple) with f faces, then there exists a p-toroid P' whose optimized graph of connection is a subdivision of the graph G.*

Proof. For a given graph G which is the skeleton of some polyhedron π, we shall form a corresponding subdivided graph G' by splitting each edge of G and adding a new node between the splitted parts. Let us mark the old nodes in gray and the new ones in black.

Starting from the graph G', we shall form a toroid P' in such a way that each of the black nodes represents polyhedra of type Π and each of the gray nodes v of G' represents polyhedra of type V_k, where k is the number of edges from v. Since the nodes v of G' were originally vertices of the polyhedron π, $k \geq 3$ is always satisfied. Pieces of type Π and V_k can be connected in the following way: if $A_1, A_2, \ldots, A_k$ are vertices in the basis of V_k and V is the apex, then the contact faces of V_k, $k \geq 3$ and one of the bases of neighbouring polyhedra of type Π would be $A_i A_{i+1} V$, $i \in \{1, \ldots, k-1\}$, $A_k A_1 V$. If necessary, either the polyheda Π or V_k could be slightly deformed; especially polyheda Π could be with skew placed bases.

In this construction, the faces of the polyhedron π will be transformed into handles of P' because the inserted prisms Π allows the pyramids V_k to be far enough apart to form them. So, graph G' is an optimized graph of connection for toroid P'.

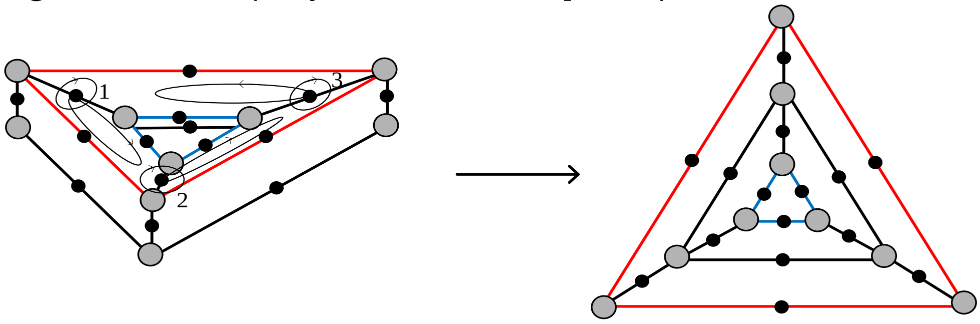
Note that the graph G' is visually similar to the toroid P', because the prisms Π looks like strings, and thus represent edges of P', while the pyramids V_k represent the vertices of P'. This is the reason why in the figures that illustrate the following examples we shall use the graph G' instead of the corresponding toroid P' and follow in parallel what happens to the mutually corresponding toroid P' and graph G'.

To calculate the number of tetrahedra in 3-triangulation, we mention that Π has 6 vertices and is 3-triangulable with 3 tetrahedra, while V_k has k+1 vertex and is 3-triangulable with $k - 2$ tetrahedra. Counting the vertices of P', it is sufficient to consider only the vertices of the pyramids of type V_k, because all the vertices of the prisms Π belong to some of the contact faces.

Example 1

We shall start with the 1-toroid P_9 given in Figure 2 as a polyhedron π. Let us denote with h the number of handles of the starting polyhedron. Since P_9 is a 1-toroid it holds h = 1. We shall then construct the corresponding graph G' and toroid P', as in the Theorem 2. For each of the 9 vertices of P_9, there are 4 edges that are incident to it, so we shall use V_4 as a component corresponding to the gray nodes of the graph G'. Since P_9 has18 edges, we shall use 18 prisms Π obtained form black nodes of G'.

Figure 4. *Skeleton of Polyhedron* P_9 *as a Graph* G' *of 10-Toroid*



Considering the number of handles for the obtained toroid P', we shall do more cutting in accordance with the definition of p-torus. The first 3 cutting are shown on the left side of Figure 4. After that, the only handle of P_9 will lead to two boundaries, i.e. two cycles of G'. The inside one is marked in blue and the outside in red. Then we have the situation as on the right side of Figure 4. We can see that it would be necessary 7 more cuttings to get a graph without a cycle or a corresponding polyhedron without handles. So, the constructed toroid P' has p = 10 handles.

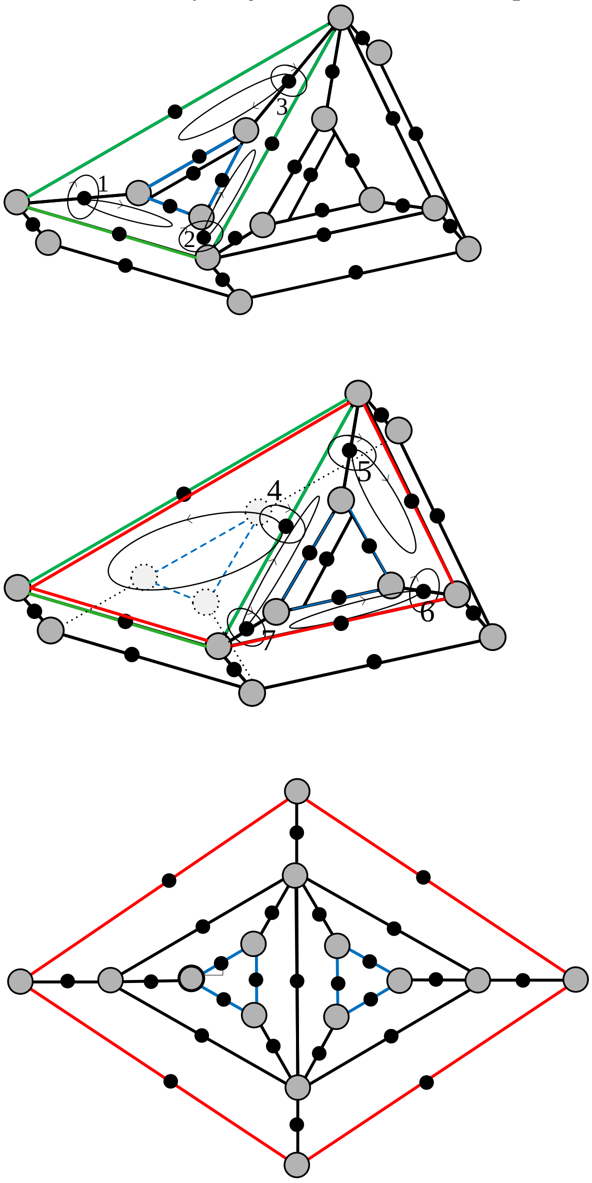
The number of vertices of P' is $n = 9 \cdot 5 = 45$, while the number of tetrahedra in the 3-triangulation is $T = 9 \cdot 2 + 18 \cdot 3 = 72$. Theorem 1 guarantees for P' that $T_{\min} \geq 45 + 3(10-1) = 72$. Thus, in this case, the lower limit given in Theorem 1 is reached.

We can conclude that the faces and the two boundaries obtained from the handle of P_9 lead to the handles of P' excluding "outer face". Since the number of faces of P_9 is f = 9, h = 1 and the number of handles of P' is 10, it holds $p = f + 2h - 1$.

Example 2

The 2-toroid P_{14} will be used as a polyhedron π, i.e. $h = 2$ in this example. For P_{14} it is also valid that it has 14 vertices, whereby 10 of them are incident to 4 edges and 4 of them are incident to 5 edges. So, the 10 components that correspond to the gray nodes of the graph G' will be pyramids V_4 and 4 of them V_5. There are 32 edges of P_{14} leading to 32 prisms Π. Note that for P_{14} the number of faces $f = 16$.

Figure 5. *Skeleton of Polyhedron P_{14} as a Graph G' of 19-Toroid*



In the cutting process, the first 3 of them are shown in the first part of Figure 5. As in the previous example, after this part of process, two boundaries will remain, marked in blue and green. The next 4 cutting are shown in the second part of Figure 5 with indicated previous green and dashed (invisible, behind green cycle) blue boundary. The green cycle gives the next cut and will not appear in the last part of process. It is also indicated that two blue ones remain - inner boundaries, the old and the new one, and also one red - outer boundary. In the end, there will remain a planar graph with 12 cycles (and one "outer", which is not taken into account). There are $p = 19$ handles of the new toroid P'. The number of handles p is also obtained by taking the number of faces f and two times the number of handles h of P_{14}. So, again is true $p = f + 2h - 1$.

Calculating the number of vertices of P' we obtain $n = 10 \cdot 5 + 4 \cdot 6 = 74$ and the number of tetrahedra in 3-triangulation is $T = 10 \cdot 2 + 4 \cdot 3 + 32 \cdot 3 = 128$. By Theorem 1, the lower limit for the number of tetrahedra is $T_{\min} \geq 74 + 3(19-1) = 128$ and it is reached again.

Conclusions

Using the concepts of piecewise convex polyhedra and of graph of connection, the properties of 3-triangulations for *p*-toroid, if any, are investigated. Based on the given graph as its connection graph, a *p*-toroid is constructed. Also, two examples of graphs and corresponding toroids are given. In the both cases, the considered graphs are skeletons of toroids. So, it was interesting to determine how many handles the toroid induced by the mentioned construction would have.

For *p*-toroids the minimal required number of tetrahedra for 3-triangulation is important property. That was the reason to examine also that number in the given examples. The result is that the lower limit obtained in the previous papers of the author has been reached.

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On the Determination of the Geographic North on Archeological Plans in Connection with the Problem of the Quality of Geographic Education

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The publication is dedicated to the memory of N.A. Bogdanov (1954-2020), Doctor of Geography, Leading Researcher of the Laboratory of Geomorphology of the Institute of Geography of the Russian Academy of Sciences, who participated in the preparation of materials and publication of abstracts on March 1, 2020. The article raises the question of the need for the development of universal geographic education, which forms the cultural appearance of a person and the scientific picture of the world of our era. The consequences of typical technical errors in working with the topographic plan are analyzed: the sides of the horizon are not indicated; north is specified with an error; the original high-quality materials were used incorrectly; magnetic north is indicated, but the year of the topographic survey is not indicated; supplemented topographic maps are superimposed on the historically previous topographic base. The changes in the current situation, according to the authors, are facilitated by the following tasks: to increase the number of hours of teaching geography at school; pay more attention to practical orienteering exercises; expand the selection of popular geographic literature to raise public awareness of the possibilities of classical methods and fundamental achievements of modern geography; and some others.

Keywords: *cartography, education, archaeological sites, interpretation, information*

Introduction

Interdisciplinary geo- and astro-archaeological studies of recent decades have shown that cultural heritage objects oriented along the geographic meridian can be investigated as the functions of the simplest astronomical instruments. Comprehensive studies of the orientation of artificial and natural-man-made objects make it possible to obtain a significant amount of additional information about a person in the nature-culture system during the period of anthropogenesis. This information is also needed to clarify the modern natural science picture of the world, to improve the models of the Earth's evolution. It includes processes and characteristics such as planetary parameters (inclination of the earth's axis, illumination mode and solar climate, change in gravity); geological and

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geomorphological conditions (speed of tectonic uplifts, intensity of denudation, etc.), as well as changing climatic conditions.

Prospects for the use of information stored in the structure of prehistoric objects are associated with the creation of a database. It will allow correlating information on objects and taking into account geographical positioning (coordinates, etc.).

The materials accumulated in classical archeology can play a significant role in creating such a database. However, there are a number of problems that complicate the use of archaeological information in complex interdisciplinary research. The most typical of them are: 1) errors in determining the geographical north made in the field descriptions of objects; 2) lack of indication of the year of topographic survey: does not allow making a correction for magnetic declination; 3) the tradition of maintaining the orientation of all subsequent plans corresponding to the first image in the study of the object.

The neglect of such an important characteristic for the geographic positioning of an object (location relative to the geographical north) is explained by the insufficient sensitivity of the methods of classical archeology to this parameter.

As a result, the useful information contained in these objects remains out of sight of most researchers. An artificially created error is the basis for widespread dissemination of the opinion about the chaotic orientation of ancient objects and the low level of astronomical literacy of their creators.

Cultural Heritage and Geographical Culture: Research Methods

In geography, cultural heritage objects are considered as elements of the geo-cultural space, which, according to V.N. Streletsky, is the integrity of nature and culture, created by a continuous flow of matter, energy and information (Streletsky 2005). As a result of a long-term interaction, all elements of this system develop: nature is saturated with natural-human-made and artificial objects necessary for human life, and culture (material and non-material) is saturated with models of nature, which most accurately convey the vital connections of the surrounding world (Paranina 2020). The concept the world around, like the subject of the same name in the system of primary school education, covers geo-space (the geographic shell of the Earth, including elements of the socio-sphere), objects of the Solar System and the Universe. This approach is the basis for the formation of systems thinking in childhood, a resource for increasing the awareness of any practical actions in adulthood, as well as a guarantee of responsibility in the field of managerial decision-making (Mazurov 1999, Solomin 2001, Dronov et al. 2018, Grigoryev et al. 2020). The worldview role of geographic representations is one of the important reasons for the development of geographic education, which could become continuous and universal.

The most important component of the geographical culture of V.P. Maksakovsky called the ability to use the language of geography that represents a system of signs used to express inferences, concepts, laws, theories of geography (Maksakovsky 1998). According to educators, the components of the language of geography are: concepts and terms; facts, figures and dates; geographical names and language of the map; geographical representations (images) (Tamozhnyaya et al. 2016). In this article, we will try to analyze the influence of geographic education and geographic culture on the quality of scientific products created in the fields of modern humanitarian knowledge. It is difficult to overestimate the severity of the problem, since these concepts shape the mentality and perception of the world.

In modern scientific literature, you can find a wide variety of world models of ancient man, developed by specialists in different scientific fields. Humanitarian studies of cultural heritage are focused on the artistic value and aesthetic qualities of heritage sites and form an idea of the mythopoetic model of the world of ancient man. In the geography of culture, the subject of research is all forms of over-biological adaptation to the environment, and the result of the research is the concept of information modeling of the world, based on the measurement and designation of parts of geographic space and time (Paranina 2020, Paranina and Paranin 2017a, 2017b, 2018, 2019, 2020). Scientific discussion, to a certain extent, is due to the clash of opinions, representing antagonistic schools of thought and ideological trends (mainly the conflict between idealism and materialism). But, in many cases, the problems of understanding are associated with the lack of formation of the authors' geographical competences: poor knowledge of the fundamental laws of nature, incomplete geographical data and elementary errors in field measurements.

Results and Discussion

Typical Problems of Determining the Geographic North on Topographic Plans

Analysis of publications shows that the determination of the north in the schemes and plans may be difficult or impossible for several reasons: 1) the sides of the horizon are not indicated; 2) north is specified with an error; 3) the original high-quality materials were used incorrectly; 4. marked magnetic north, but not indicated the year of the survey; 5) supplemented topographic maps are superimposed on the historically previous topographic base. Let us analyze a few typical examples.

North is not indicated in the sketches made by archaeologists who study ancient petroglyphs as works of art (Figure 1). In the given example, a sample of the rock art monuments Lahouirra. This place of concentration of a large number of petroglyphs is located in the south of Morocco, 55 km south-south-east of Foum Zguid. The authors note that the images are replete with signs and figures of the symbolic style (spirals, snakes).

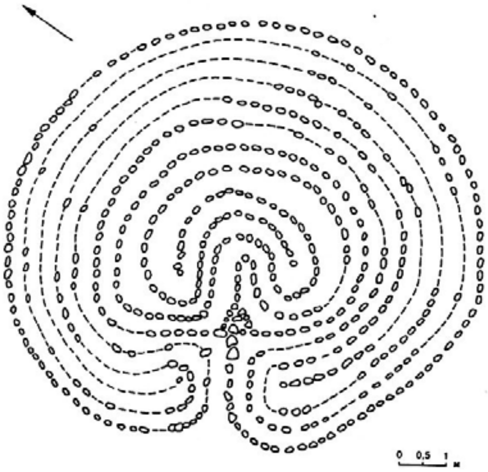
Figure 1. *Lahouirra* (Nami et al. 2007): *Photo & Drawing, North is Not Indicated*

Art analysis focuses on the similarities and differences between objects in terms of the size and shape of signs, the depth and thickness of lines. As a result of this scientific approach, the geographic (spatial) characteristics of these objects are ignored, and the huge amounts of published data on petroglyphs of North Africa remain *dumb* for the study of their rational functions. We are, in particular, interested in the navigational purpose of these objects: as markers of geographic space, tools for determining the time (sundial-calendars), geo-position indicators necessary for orientation in the landscapes of the savannah and desert in the conditions of the transit location of the territory and the nomadic economy.

An example of publications when *a direction* is indicated on the plan *does not correspond to the real azimuth indications of the geographic or magnetic north* (Figure 2a) can be taken from the scientific archaeological journal "Kizhi Vestnik" (Manyukhin 2002). Possible reasons for the error: 1) faulty compass or inability to use it, 2) inability to determine and correct for magnetic declination; 3) negligence in the design of the scientific report and publication. In the text of the article, on the basis of a mistake, the author claims that the object under study is not oriented along the sides of the horizon.

Keretsky Labyrinth was investigated twice by the authors of the article - in 2016 and 2017. The azimuth of the entry is 255º (Figure 2b), the magnetic declination on July 17 is 14ºE, hence the true azimuth is 269º, i.e., the entrance to the labyrinth is from the west (in 2017, measurements were made with a mountain compass, the result was repeated). It follows from this that the Keretsky Labyrinth figure reflects the main geographic directions, and this allows it to be used for orientation in geographic space-time. The location of the axis of the figure on the geographical parallel is convenient for determining the days of the spring and autumn equinox - the main calendar boundaries in the Arctic region, dividing the year into severe winter and summer time, highlighting the boundaries of the period of high productivity of ecosystems and marine navigation.

Figure 2. *Keretsky Labyrinth on Krasnaya Luda (Karelia): a - Plan of the Object with an Error in Indicating the Azimuth (Manyukhin 2002); b - Measurement of the Azimuth along the Line of the Entrance to the Maze (Paranina and Paranin 2017a)*



a

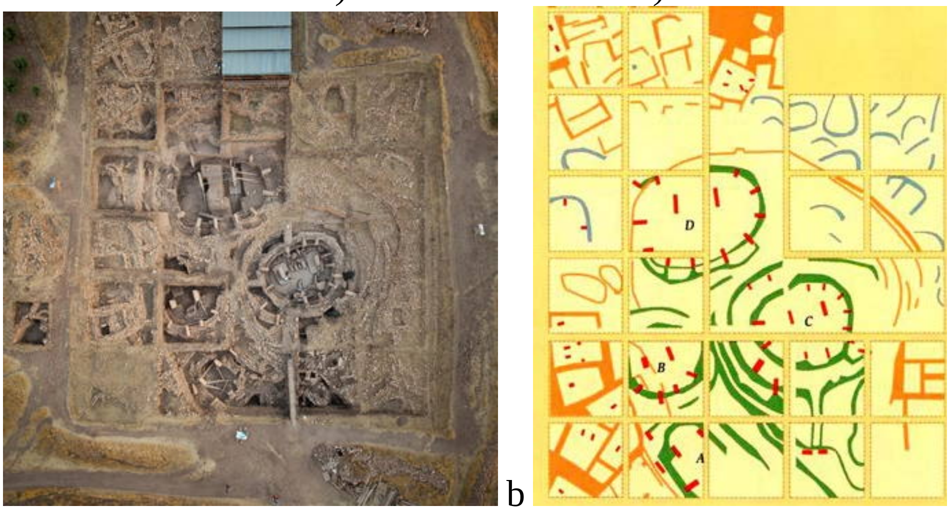
b

The question arises why the axis of the Keretsky Labyrinth (and a similar labyrinth on the Oleshin Island) was not oriented by the builders along the meridian, which would make it possible to use them as a sundial and a solar calendar along the length of the resulting line, sequentially crossing the arc? This limited function of the instrument can be explained by the fact that the small islands and peninsulas of the White Sea were used not for permanent residence, but for a short stay in the summer - in the season of intense navigation, fishing tuna and sea hunting.

The consequence of insufficient geographic education is not only the replication of errors, but also the incorrect use of quality data. Let us consider this using the example of topographic plans of the Göbekli Tepe megalithic complex, published on the UNESCO website, and two variants of their paleoastronomical interpretation, made with and without magnetic declination (Figure 3 A, B).

At archaeological excavations, topographic work is traditionally carried out with the help of a magnetic compass (azimuth circle), so the topographic plans are oriented to the magnetic north. Differences in directions to the geographic and magnetic north are clearly visible when comparing the scheme of the object and its image on the satellite image (Figure 3a, b). However, some authors, using these materials (and referring to them), forget to correct for magnetic declination before starting complex astronomical calculations, and this leads them to new errors (Collins 2013, Hale and Collins 2013, Lorenzis and Orofino 2015). For example, according to the calculations of Collins, the central steles in sector D of the Göbekli Tepe megalithic complex were aligned in the NW-SE azimuth associated with the position of Orion. However, if we take into account the magnetic declination (here 5.57 ° E + 0.31 ° on July 5, 2019, the displacement rate is 0.08 ° E per year), then it can be seen that the crossbeams of two T-shaped steles standing in the center are oriented along the geographic meridian - i.e., by the sun.

Figure 3. *Göbekli Tepe Temple[1]: a - Satellite Image; b - Archaeological Plan; c - Variants of Astroarcheological Interpretation (on the left - Magnetic Declination Was Not Taken into Account (Collins 2013), on the right - Magnetic Declination of 5°E (Paranina and Paranin 2020) Was Taken into Account)*

a  b

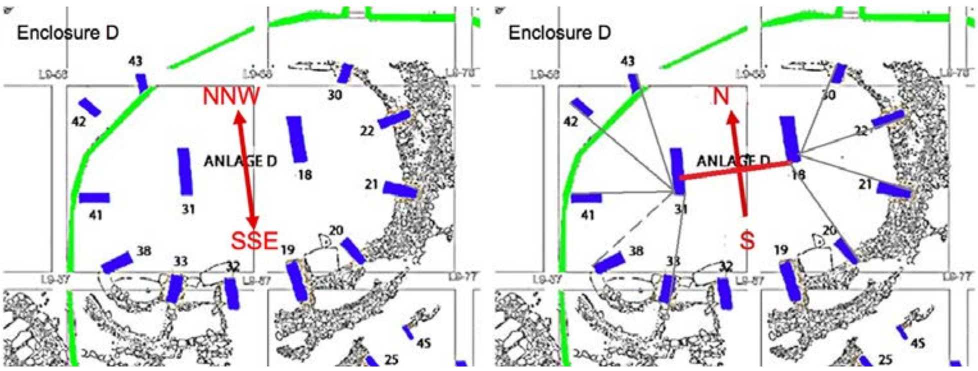
Archaeological layers:

layer 2 (8-9 millennium BC)

layer 2/3

layer 3 (9-10 millennium BC)

T-shaped columns

c 

For ease of comparison, our scheme (Paranina and Paranin 2020) is based on a topobase copied from the publication (Figure 3c, on the right). The diagram shows: the main geographic directions and positions of the shadows from the steles of the stone fence (at the moments when the shadow has the shape of a straight line). The alignment of the central steles along the geographic meridian allows you to accurately determine the *noon* (at this time the shadow of the T-shaped steles turns into parallel *straight* lines) and the *geographic north*. The use of T-shaped sundials has been noted in different regions of the world and described by archaeologists in materials devoted to the Seti I (Ancient Egypt). The location of the steles on the East-West line makes it possible to determine the days

[1]https://whc.unesco.org/en/list/1572/gallery/(3.12.2022); http://geosfera.org/uploads/fotos/gebeli2.jpg (12.01.2020)

of the equinoxes - only on these days at dawn and dawn are they connected by a shadow. In ancient times, calendars of two steles were used everywhere, because the winter/summer boundary is most important for the seasonal movements of hunters and pastoralists, the economic cycle of farmers, and the astronomical new year (vernal equinox) coincides with the general revival of nature.

It is noticeable that the T-shaped steles in the walls of the circle/ellipse stand at almost the same distance, and the lines of the direct shadow have a close angular distance - this is the basis for determining the hours (parts of the day) by the position of the Sun. The height of the Sun (and the length of the shadow) depends on the season of the year, but the angular velocity of movement across the sky is unchanged, which makes it possible to determine the hours of the day in the direction of the shadow, and the hours of the night - according to the passage of astronomical objects across the celestial sphere. The speed of movement of all astronomical landmarks is equal to the angular speed of the axial movement of our planet: 360 ° in 24 hours or 15 ° / hour (1 ° / min). In the diagram, the connections by the shadows are indicated with the help of thin lines and one dotted line - from 38 to 31 (presumably, the stele could deviate from the original direction during excavations). In the observatory under consideration, it is obvious that the seasons of the year can be determined by the time and azimuth of sunrise, determined by the shadow of several steles that form a local network. This technology naturally preceded the sundial-calendars based on the shadow of a mono-instrument - a gnomon, located in the center of the platform (Paranina 2020).

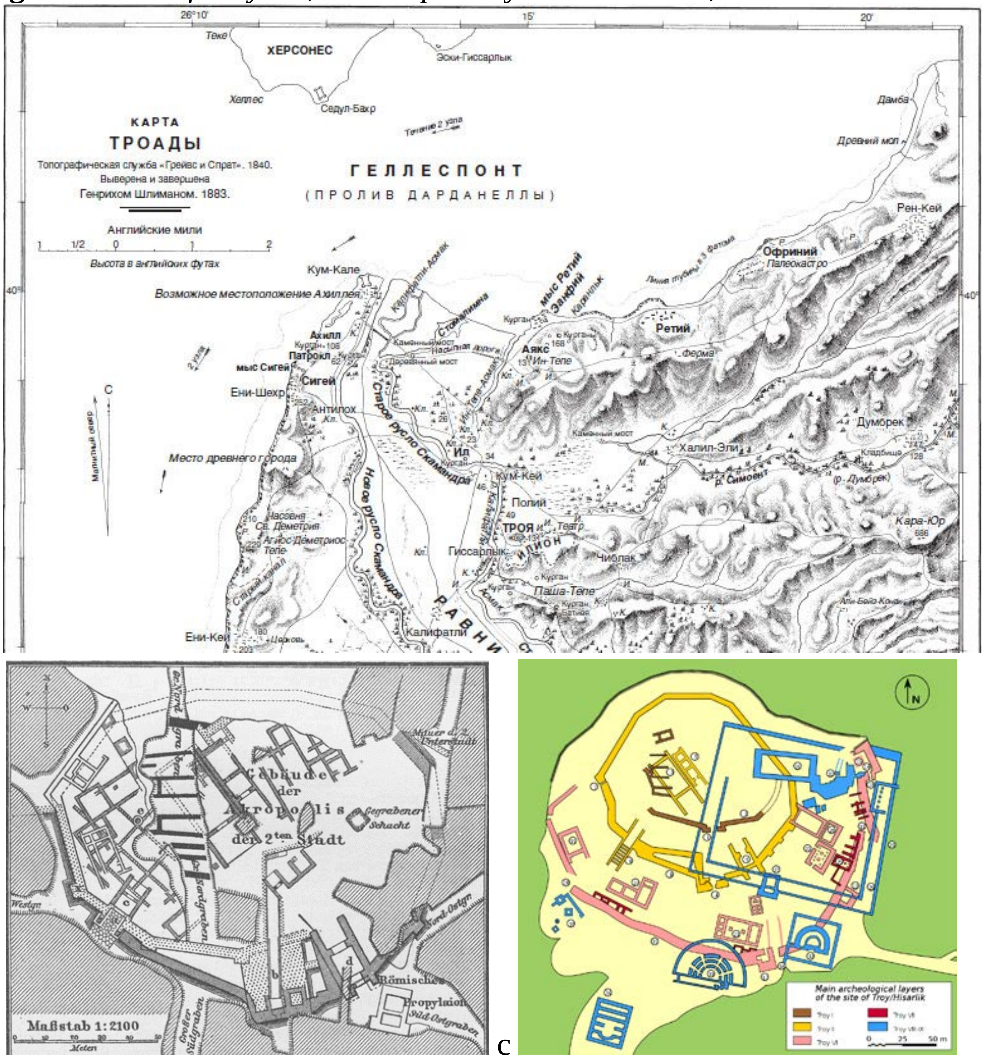
Since ancient times, sunrises in different months of the year have been associated with the zodiacal belt of the constellations. In this regard, a sighting stone with a hole on the north side of the site (between 43 and 30) could be the main element of the sidereal clock and served as a north marker necessary for building the celestial sphere and night observations. With this approach, the semantics of images on steles (petroglyphs, bas-reliefs and animal sculptures) may be associated with phenological changes in different seasons of the year, determined by sunrise in the corresponding sectors of the horizon.

Reconstruction of the calendars for sectors A, B, C, and D, plotted sequentially in the interval from 12 000 to 9 500 years ago, is of interest for understanding the relationship between the evolution of the landscape and the parameters of the planet. Differences in the direction of the central steles of the sectors are up to 15 °, which may be the basis for clarifying the dynamics of the position of the earth's axis and the rhythm of global climatic changes. However, research of the monument as a masterpiece of art and architecture now prevails, and the main task of such monumental structures is seen in the administration of religious rituals. In general, the interpretation of ancient culture is characterized by the preponderance of idealistic arguments over attempts to analyze the rational functions of objects and the development of technologies for adapting to the dynamic nature of the Earth.

A certain confusion in determining the orientation of objects is introduced by the *tradition of maintaining the orientation of the plan published by the first researcher of a large object.* The fact that it is convenient to compare the details of topographic plans from different years makes it difficult to see the real position of the object. This situation is well illustrated by a series of plans representing the cultural layers of the archaeological site - the legendary city of Troy (Figure 4). It can be seen that the direction to the north and the general orientation of the objects are the same on the Schliemann plan and on the modern plan. It could be assumed that the magnetic declination for 127 years. But this is not so: on the Troada map, Schliemann indicates the western declination, but according to the calculator, today it is eastern (2019-11-29 5.21 ± 0.32 changing by 0.11E per year).

If we take into account all the above circumstances, it becomes obvious that all objects belonging to the Hellenistic era are clearly oriented to the geographical north - Troy VIII (900-350 BC) and Troy IX (350 BC - 400 AD). And this is in good agreement with the facts of the widespread use of sundials and the associated orientation of urban objects, especially fortresses and public places - streets and theaters (blue in Figure 4C).

Figure 4. *Plan of Troy[2]: a, b - Compiled by G. Schliemann, c - Modern*



a b c

[2]https://muze.gov.tr/muze-detay?SectionId=TRO01&DistId=TRO (10.01.2020); https://whc.unesco.org/ru/list/849 (3.12.2022).

Loss of Information about the Location of Objects in the Process of their Protection and Museification

Museification measures bring objects closer to the security infrastructure, but at the same time information risks arise associated with distortion of information about the links of objects with the host geocultural space. The separation from the natural and cultural context makes it impossible to correctly interpret and fully disclose the information potential (for the study of rational purpose, semantics of form and ornamentation) (Marsadolov et al. 2019). The transfer of objects to city museums is associated with a change in microclimate and the impact of a whole range of environmentally unfavorable conditions (pollution of all environments, road dust, vibration and exhaust gases from cars). The consequence of isolation in storerooms is the impossibility of performing basic social functions by cultural objects.

It should be emphasized that only the spatial position (geographical coordinates) can give answers to questions about the role of an object in the geocultural space, about the primary rational functions of the object and the symbolism that has come down to us genetically related to them. Practice shows that it is precisely the spatial reference of objects that is lost during museification of objects, because only the administrative region, settlement or large natural object associated with its location is indicated. Moreover, in the presence of a topographic plan, the direction to the magnetic pole is shown without indicating the year and magnetic declination at the time of the topographic survey. Therefore, museums, as far as possible, need to fix the exact coordinates of heritage sites, place photographs of the surrounding landscape, as well as space or aerial photographs of their places of origin in the exposition, and leave a copy of the object or a commemorative plaque in the landscape.

Conclusions

The examples given show that the problem of geographical culture, indicated in the title of the article, has no boundaries: errors and inaccuracies can be found in official scientific journals and on author's websites, in Russian-language literature and in foreign publications. The existing practice of protection and museification makes a significant contribution to the aggravation of the problem of geographically incorrect representation and interpretation of objects.

As a result, in the scientific and popular science literature, addressed both to the specialists and to the general reader, a lot of errors are replicated, including the opinion about the randomness in the location of archaeological sites. From scientific publications, errors are copied into textbooks of schools and universities, and significantly affect our ideas about the history of domestic and world culture.

A change in the current situation, according to the authors, is facilitated by the formulation and solution of a number of tasks, it is necessary:

- to increase the number of hours of teaching geography at school; pay more attention to practical orienteering exercises;
- to deepen the level of teaching the course of General Geography for humanitarian specialties;
- to expand the selection of popular geographical literature to raise the level of public awareness of the possibilities of classical methods and fundamental achievements of modern geography;
- to improve GOSTs (Russian National Standard) and rules for the design of scientific reports and publications;
- to optimize the rules for the protection and museification of objects, taking into account the value of information about their position in spatial systems.

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Snow Height and Snow Water Equivalent Estimation from Snow Cover Fraction Using Sentinel-2 Satellite Images in North Kazakhstan

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Climate change's influence on snowpack can significantly affect natural and anthropogenic processes. Water resources and agri-business, which depend on winter precipitation, are highly affected by variations in a snowbank and melting regimes. This paper demonstrates the comparison results of the snowpack thickness estimation in the LLP "North Kazakhstan AES" adopting distinctive techniques (quadratic, exponential, and linear functions) for assessing Fractional Snow Cover (SCF) and demonstrating Snow Water Equivalent (SWE) on the one hand, and in-situ perspective on the other. Between the 26–29 of February 2020, a field measurement was managed on the territory of 25,000 hectares. Accordingly, the thickness of the snowpack was surveyed at 560 points, and its density was measured at 70 points. Applying existing methodologies of SCF computation, it became apparent that the quadratic equation provides more reliable results at RMSE of 0.01 m, followed by linear -0.12 m and exponential -0.13 m methods. This work showed a strong correlation between snow height and SCF, namely the quadratic function in Northern Kazakhstan. Thus, we strongly suggest using Sentinel-2 MSI and the quadratic SCF estimation function for snow cover estimation, further spring flood forecasting, and other hydrological studies.

Keywords: *snow height, snow cover fraction, normalized-difference snow index, snow water equivalent, Sentinel-2 MSI, North Kazakhstan*

Introduction

Due to the high natural spatio-temporal variability of the snow cover and its quick directional changes caused by the changing climate, the development of techniques for acquiring accurate information on the snow cover with high spatial and temporal resolution across broad areas is crucial (Armstrong and Brun 2008). Snow cover impacts nearly all processes of interaction of the atmosphere with the underlying surface of temperate and high latitudes in the cold season because of thermophysical solid properties, the significant fluctuation of parameters, and the

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duration of occurrence across broad land regions (Varvus 2007). An important influence on natural and artificial systems may result from the effect of global warming on snow cover (early snowmelt, shorter period with stable snow cover). Since the snow cover in the mountainous and lowland areas of the temperate climate zone is more subject to temperature fluctuations, snowmelt is most likely to increase in these areas in the near future (Popova et al. 2015). The water resources (changes in the intensity of spring floods, greater capacity for evaporation) and the agriculture sectors that depend on them are impacted by variations in the regime of snow accumulation and snow melting (Barnett et al. 2005).

The soil and climatic conditions of the region of Northern Kazakhstan are ideal for the growth of cereals, oilseeds, legumes, and forage crops, and, in particular, for the production of food spring soft wheat with high content gluten, which is in high demand on global markets due to its property of enhancing the baking properties of flour. Peas, lentils, rapeseed, sunflowers, flax, spring wheat, millet, and oats occupy a significant part of the land (Shur 2014). Because of its position, the North Kazakhstan region's agricultural fields receive natural irrigation from rainfall and moisture stored in the soil from snowmelt. To get the best harvest of farm fields, particularly winter crops, it is necessary to know how much snow was melted in a particular location and whether there is enough moisture in the soil.

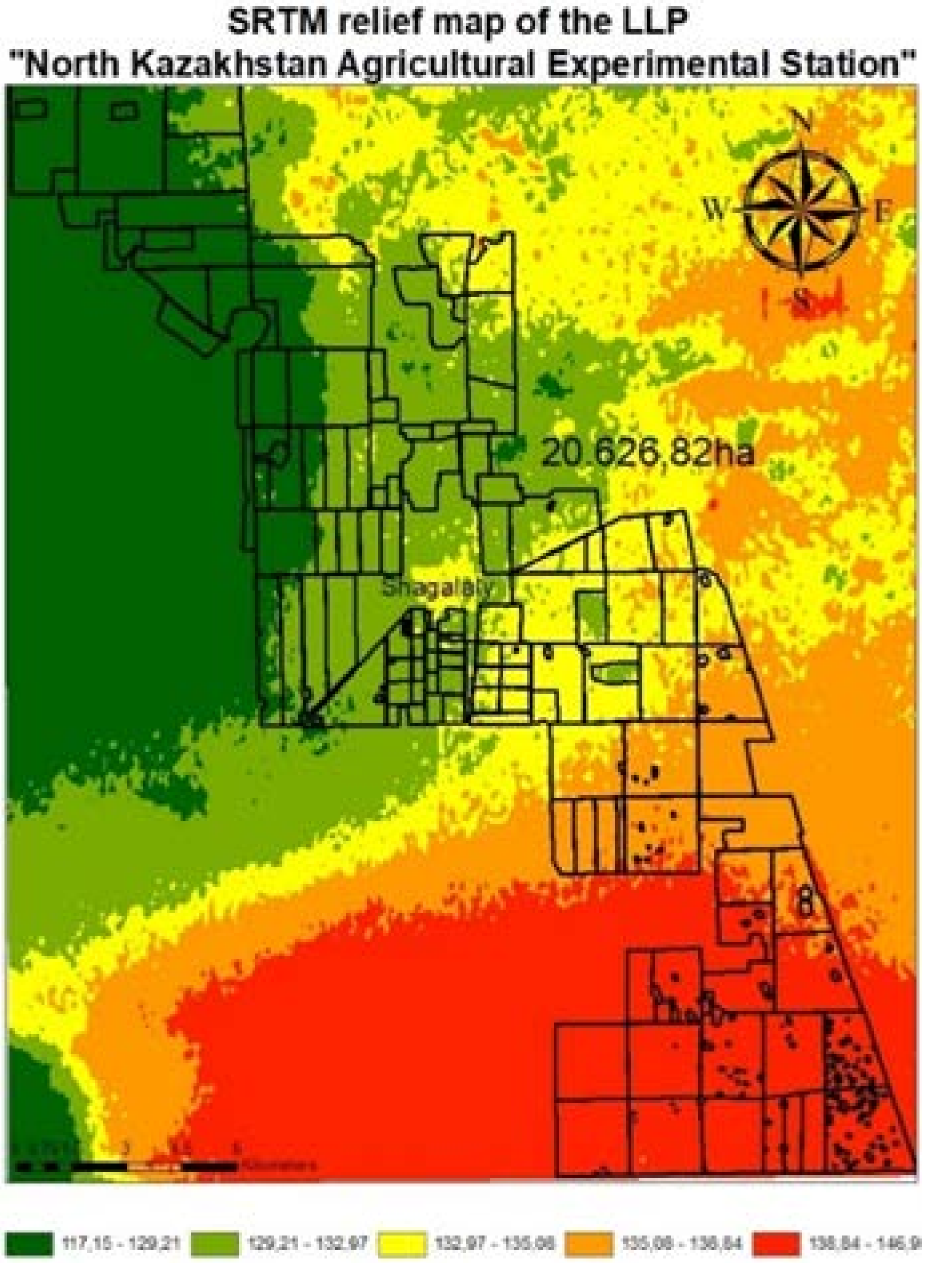
It is known that the thermal conductivity and density of the snow cover increase with increasing snow depth. Loose, recently fallen snow has the lowest heat conductivity and has the most significant insulating impact on winter crops. On the other hand, compacted snow offers less protection from freezing to winter crops. Since a decrease in soil temperature at this depth up to a certain point damages the tillering node and frequently results in plant death, the question of how snow cover affects soil temperature at a depth of winter crops' tillering nodes (3-5 cm) is of particular interest to the agricultural industry. The soil temperature at the level of tillering node, therefore, turns out to be 1-3°C higher than the air temperature in the presence of a snow cover between 1 and 5 cm thick (Ventskevich 1952). The average thickness of the snow cover in Kazakhstan's northern areas is between 40 and 70 cm, according to the "Map of the Depth of Snow Cover" (Richter 1948).

To determine the safety of winter crops in crucial fields of the North Kazakhstan region in the third decade of February 2020, this study will estimate the snow height (SH) and snow water equivalent (SWE) from SCF maps derived using NDSI and different regression equations. The depth of water that may occur if the snow were to melt entirely is known as the snow water equivalent (SWE). It can describe a restricted snow pattern over a matching specified surface area or snow cover over a specific region. The snow water equivalent is calculated by multiplying the vertically integrated density in kilograms per cubic meter by the snow depth in meters (Goodison et al. 1981).

Study Area

The research was conducted in the territory of the agricultural enterprise "North Kazakhstan AES," located in the Aqqayin district in the North Kazakhstan region (54° 17′ N, 69° 52′ E). On the 72 thousand hectares shown territory, of which the arable land and agricultural fields cover almost 25 thousand ha, the study area's characteristics include relatively level relief and the existence of several lakes (see Figure 1).

Figure 1. *DEM of the LLP "North Kazakhstan AES"*



Source: Compiled by the authors.

According to the Gismeteo data, the southwest wind was predominant throughout the field snow survey, reaching a maximum speed of 13 m/s and an average value of 6.4 m/s (Gismeteo 2020). A preponderance of south-southwestern points was another characteristic of the wind direction for the cold spell of 2019–2020. The direction of the wind is of great importance as it affects the thickness of the snow cover; snow from the tops of the hills is blown away by the wind and accumulates in depressions and ravines at the base of the slopes.

Materials and Methods

Snow height, density, Snow Cover Fraction (SCF), and Snow Water Equivalent (SWE) may be measured both in-situ and via remotely sensed data. The snow-water equivalent is a crucial cryospheric research parameter since snow, water, and ice are essential to agriculture and the climate system.

Ground-Truth Field Survey

From February 26 to February 29, 2020, a field snow survey was conducted at prominent locations. Over the area of LLP "North Kazakhstan AES," the snow cover was assessed at 560 points (snow height and density). A metal ruler with a 1-mm scale was used to measure the thickness of the snow, and the weight snow gauge VS-43 was used to determine the snow's density. We employed a snowmobile that belonged to the S. Seifullin Kazakh Agrotechnical University's scientific and educational center of GIS technologies for rapid and easy mobility across the research region (see Figure 2). The mass (m) of measured snow and the height (SH) of measured snow were used to compute the actual density (ρ) of snow:

$$\rho = \frac{m}{10 \times SH} \tag{1}$$

where ρ is in kg/m^3, SH in m, and m in kg.

Figure 2. *Measurement of Snow Height and Density Using a Weight Snow Gauge (VS-43)*

Source: Compiled by the authors.

However, the literature has several approaches for calculating snow density with minimal input parameters. We especially want to highlight Sturm et al.'s (2010) study, which offers a variety of highly straightforward solutions for snow density that are accessible through a single equation:

$$\rho = \rho_0 + (\rho_{\max} - \rho_0) \times (1 - \exp\left(-k_1 \times HS - k_2 \times \frac{DOY}{100}\right)), \quad (2)$$

where variables ρ_0, ρ_{max}, k_1 and k_2 are accessible in Table 1, and DOY is a simple day of the year which starts with -92 on October 1 to -1 on December 31 and from 1 on January 1 till 181 on June 30.

Table 1. *The Set of Variables of Snow Classes for the Equation 2*

Snow class	P_0	P_{MAX}	K_1	K_2
Alpine	223.7	597.5	0.12	0.38
Maritime	257.8	597.9	0.10	0.38
Steppe	233.2	594.0	0.16	0.31
Tundra	242.5	363.0	0.29	0.49
Taiga	217.0	217.0	0	0

Source: Sturm et al. 2010.

For the Snow Water Equivalent (SWE) calculation, a simple formula was utilized:

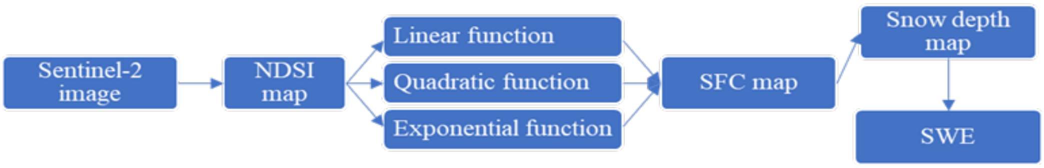
$$SWE= \rho \times SH. \quad (3)$$

where SWE is in kg/m^2, ρ in kg/m^3 and SH in m.

Remote Sensing Data

We took the multispectral satellite images of Sentinel-2 MSI with the product ID: S2B_MSIL1C_20200308T062709_N0209_R077_T42UWF_20200308T092518 and S2B_MSIL1C_20200308T062709_N0209_R077_T42UWE_20200308T092518 and performed the whole data processing flow to compute the snow height (SH) and snow water equivalent (SWE) from Normalized-Difference Snow Index (NDSI) (see Figure 3).

Figure 3. *Flow Chart for Data Processing Methods*



Source: Compiled by the authors.

The NDSI is based on the variation in how well snow absorbs light in the visible and infrared spectrums. Thus, the method is only used during the day, evening, or night. Pixels coated in ice will not be seen (Kim et al. 2017). The difference between the reflectance of light from Sentinel-2 with wavelengths of 560 nm (Band 3) and 1610 nm (Band 11) is used to construct the NDSI index.

$$NDSI = \frac{B_3 - B_{11}}{B_3 + B_{11}}. \quad (4)$$

where B3 is the Green band, and B11 is the SWIR band of Sentinel-2 MSI. Once the Normalized-Difference Snow Index is calculated, we can define the area covered by snow as having pixel values higher than 0.4.

Calculating the Snow Cover Fraction (SCF) within each picture pixel is the most crucial step in modeling the snow height using RS data. It ranges from 0 to 1 and shows the percentage of the snow-covered pixel (0 percent - 100 percent). Three NDSI SCF estimate formulae are currently available in the literature:

1. linear function (Salomonson et al. 2004)

$$SCF = a + b \times NDSI. \quad (5)$$

where a and b are constants equal to -0.69 and 1.91, respectively.

2. quadratic function (Barton et al. 2000)

$$SCF = a + b \times NDSI + c \times NDSI^2. \quad (6)$$

where a. b, and c are optimized constants equal to 0.18. 0.37 and 0.25, respectively.

3. exponential function (Lin et al. 2012)

$$SCF = a + b \times e^{c*NDSI}. \tag{7}$$

where a. b and c are equal to -0.41. 0.571 and 1.068, respectively.

Romanov and Tarpley (2004) discovered a high correlation between SCF and SH during the calculation of snow height (SH), as the snow height rose with increasing Snow Cover Fraction. They have suggested the following equation based on this observation:

$$HS = e^{a*SCF} - 1, \tag{8}$$

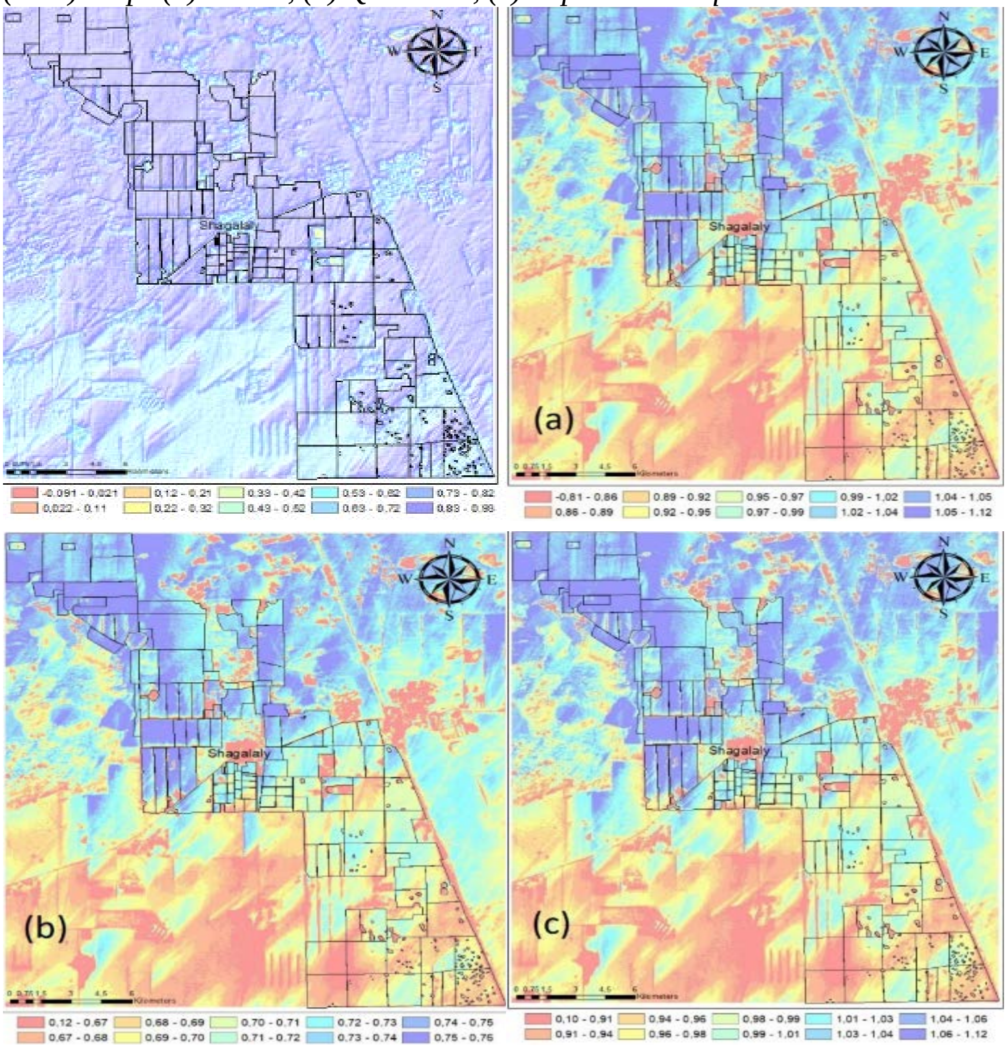
where a is equal to 0.33.

The SH was estimated using the formulas (7) and (8); then, they were validated by ground-truth snow surveying results. Furthermore, we calculated the SWE using equation (3).

Results

Using the linear, quadratic, and exponential equations as stated before, Figure 4 demonstrates the Normalized-Difference Snow Index (NDSI) and Snow Cover Fraction (SCF) maps of the study area. Due to the weather conditions in Northern Kazakhstan and the DOY when the image was taken, 99 percent of the research area was covered in snow. Therefore, there was no need to mask the NDSI map. The results from the exponential and linear equations are almost identical. However, the figures for the snow percentage from the quadratic equation are relatively low. Nevertheless, the three SCF maps' patterns do not differ much.

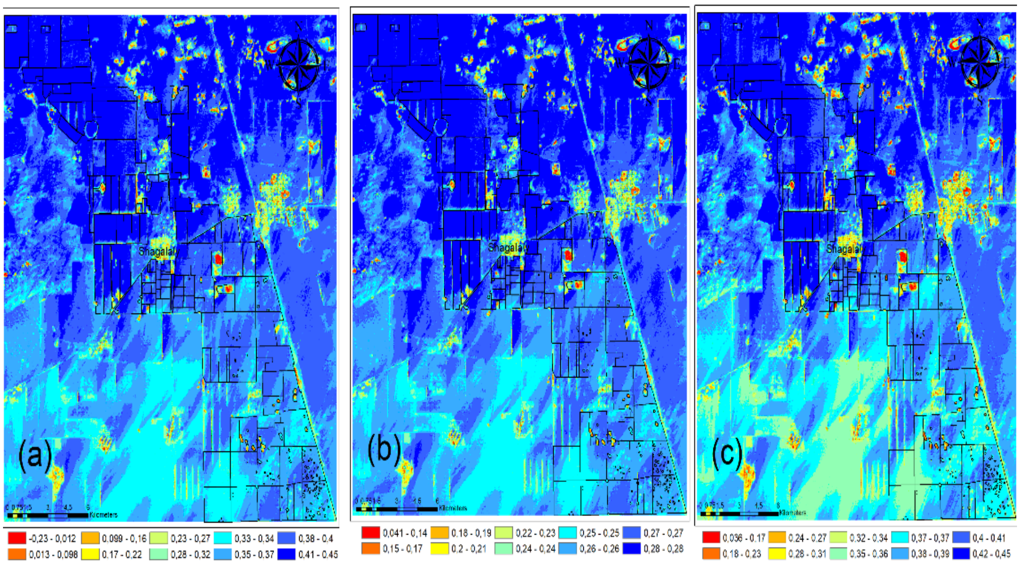
Figure 4. *Normalized-Difference Snow Index (NDSI) and Snow Cover Fraction (SCF) Maps (a) Linear, (b) Quadratic, (c) Exponential Equations*



Source: Compiled by the authors.

The equation (8) for each SCF calculation method, including the linear, quadratic, and exponential SH equations, was used to create the snow height maps (see Figure 5). The variances are more pronounced since the same algorithm was used to estimate the snow depth for each image. The maximum snow height for the SH maps based on the exponential and linear formulae was around 0.45 m, whereas the maximum snow height for the quadratic equation was only about 0.28 m.

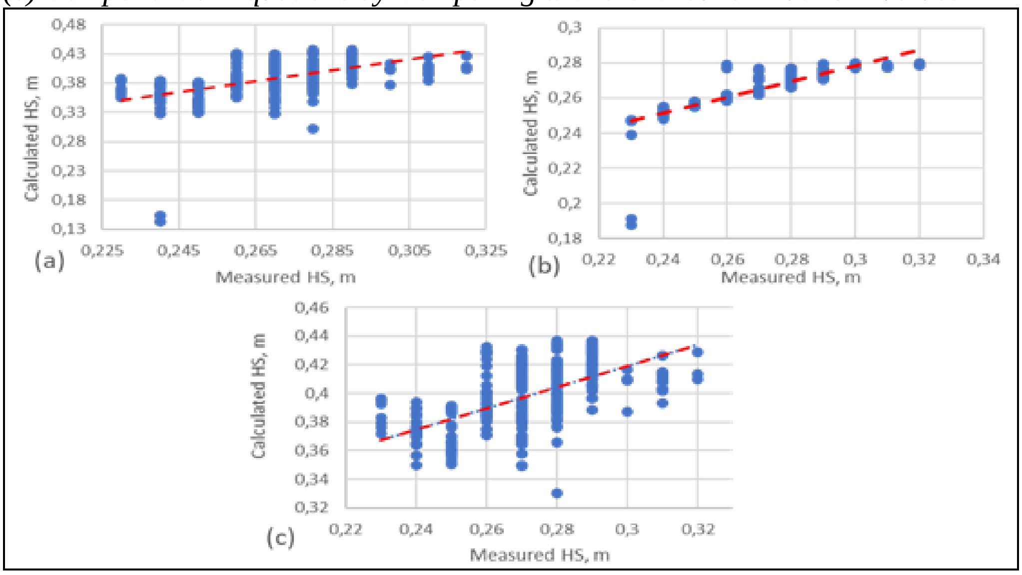
Figure 5. *Snow Height (SH) Maps Generated Using (a) Linear, (b) Quadratic, and (c) Exponential Equations of SCF*



Source: Compiled by the authors.

With a root-mean-square error (RMSE) of 0.12 m for linear equations, 0.01 m for quadratic equations, and 0.13 m for exponential equations, we further showed the correlation between the ground-truth snow height and modeled snow height maps (see Figure 6). Due to the SCF's negative representation, the snow height frequently has a negative value in the linear equation. Since the variables were overestimated at the SCF level calculation, the snow height value is often low in the quadratic function. In contrast to the other two situations, the data in the exponential approach is typically significantly more confusing.

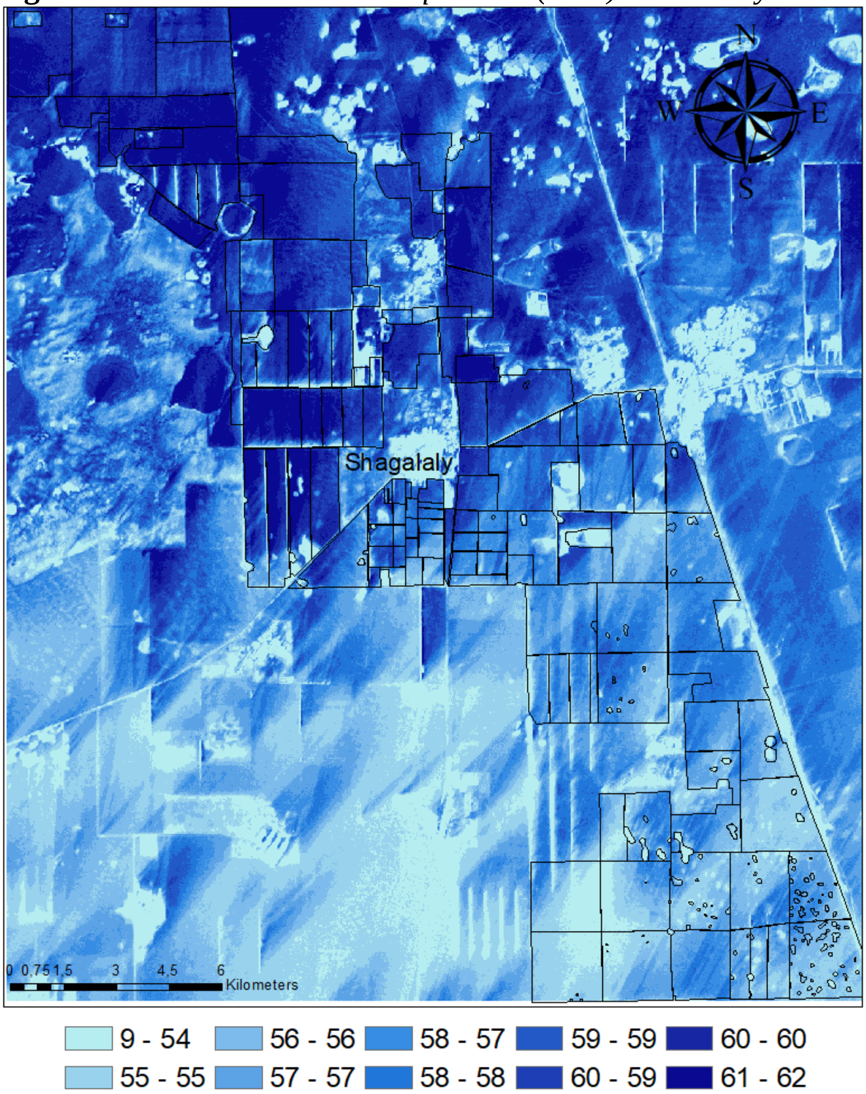
Figure 6. *Validation of Simulated SH Maps with (a) Linear, (b) Quadratic, and (c) Componential Equations by Comparing with the Ground-Truth SH Values*



Source: Compiled by the authors.

Overall, based on the statistical data, we chose to use the SH map produced using the quadratic function of SCF calculation to determine the final Snow Water Equivalent (SWE). Equation (3) was used to calculate the result, and the quadratic equation's snow height (SH) was multiplied by the average observed snow density (ρ_{avg}=216 kg/m^3) (see Figure 7). It is evident from the map that the southwest wind impacted the amount of snow that accumulated behind the objects on the opposite northeast side because of the swirl and in the low-elevation places. Additionally, the southern agricultural fields had at least 1 cm less snow cover, while the low-lying northern area of the LLP "North Kazakhstan AES" preserved substantially more snow.

Figure 7. *Modeled Snow Water Equivalent (SWE) in the Study Area in kg/m^2*



Source: Compiled by the authors.

This study applied the recommended SCF calculation technique to a straightforward Sentinel-2 multispectral image using NDSI, and three different snow height (SH) maps were produced from the estimated SCF. Several factors affect SCF and NDSI, although NDSI and SCF have a reasonable correlation. Additionally, each method's formula is specifically designed for study fields other than the region of northern Kazakhstan. Moreover, it should be noted that in the metropolitan, built areas, the snow disappears because of the snow removal even if there is a measured in-situ snow height (SH); thus, there can be an error.

Conclusion

We applied three different regression methods of calculating the Snow Cover Fraction to data from the LLP "North Kazakhstan AES" to simulate snow height using remotely sensed images and NDSI. The results were compared as a first step toward developing an optimization to measure the snow depth in the study area. The actual ground-truth snow height was used to verify the modeled snow height. The quadratic formula (0.01 m) produced the best results, followed by the linear (0.12 m) and exponential (0.13 m) equations in terms of RMSE. The lowest snow level was predominantly found inside built-up areas, and settlements on the SH map were produced using a quadratic equation and accounted for 4-5 cm. The study area's northernmost region, where the elevation was lowest according to the SRTM DEM, had the thickest snow cover, which was 28 cm. The average snow depth in agricultural fields was 20 cm, which is more than safe for winter crops because even 3-5 cm of snow may raise the temperature of seeds by 1-3°C in comparison to the air, and this quantity of snow, when melted, can add enough moisture to the soil to make it very fruitful. According to the produced SWE map, the lowest water level recorded was 8 mm, the average was 45 mm, and the maximum level was more than 60 mm. At least for the Northern Kazakhstan region, we can confidently assert that the quadratic equation of the SCF calculation suits the best, and it offers a fantastic potential to produce trustworthy SWE maps.

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