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Heat Storage as Evidence of Hydrographic Cycles in the Southeastern Mediterranean Basin

By Ibrahim A. Maiyza[*], Tarek M. El-Geziry[±] & Shimaa I. Maiyza[°]

Numerous studies were conducted in the Southeastern Mediterranean Sea to determine whether variations in the hydrography and fisheries exhibit cyclic behaviour. This work investigated the long-term cyclic behaviour of physical properties in the Southeastern Mediterranean basin; taking heat storage as a parameter of interest. Heat storage is said to be a more accurate component to measure the probable thermal cyclic behaviour within a specific basin; it allows for the elimination of diurnal (full) and monthly effects on the examined thermal behaviour to a lesser extent. The work aims to: (1) provide the best fit model of heat storage anomaly changes; and (2) investigate any cyclic behaviour of change over a considerable span. The hydrographic data (temperature and salinity) were scattered over the period 1889-2021. However, because of the scarcity of data over the period 1892-1964, calculations of the mean annual heat storage and mean annual heat storage anomaly focused on the period 1965-2021 of continuous records. The minimum mean annual heat storage anomaly of the 300 m layer ranged from -9.0E9 Jm^{-2} (1992) to +0.84E9 Jm^{-2} (1994), with an overall mean of -0.016E9 Jm^{-2}. +0.43E9 Jm^{-2} was the standard deviation from the mean. The best fit model was produced for the variations in the mean annual heat storage anomaly. This was represented by a cubic equation with R^2 of 0.21. The minimum anomaly occurred in 1980 and the maximum in 2015. Therefore, the results confirmed the 70-year cycle of variation in the hydrographic conditions in the south-eastern Mediterranean region.

Keywords: *Southeastern Mediterranean Sea, heat storage, anomalies, cycles*

Introduction

A considerable body of scientific evidence reveals that the Earth's atmosphere has changed over time, indicating that the planet is warming (Masson et al. 2021, Thapa et al. 2021). This change is primarily caused by manmade activities that emit pollutants, with natural causes having a minor role (Jonathan and Raju 2017). Many signs of climate change can be found all around the world, e.g., higher air temperatures, drier locations, wilder weather, warmer seas, and faster rates of sea-level rise (Tonbol et al. 2018, El-Geziry 2021). According to the 5th Assessment Report of the Intergovernmental Panel on Climate Change (IPCC 2013), global surface temperature data calculated using linear trend showed a 0.85°C warming

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from 1880 to 2012. The spatiotemporal distribution of the increase in global air temperature is not uniform over Earth. The increase in air temperature varies by region, with the Northern Hemisphere having the greatest increase (Bačević et al. 2020). Rising air temperatures may have various repercussions on different parts of human life, including human settlements, agricultural goods, energy consumption, environmental and social processes, and so on (Piticar and Ristoiu 2014, Bonacci et al. 2021). However, in contrast to the considered continuous global warming concept, there is another sizeable group of researchers declared that the Earth, over ages, exhibits several recognized climatic cycles and teleconnections (Ilyes et al. 2022). By definition, cycles can have a recurring impact on the Earth's climate, whereas teleconnections can have a widespread impact on the weather (Ilyes et al. 2022). Typically, temperatures rise throughout the day and fall at night. Summer and winter have warmer and colder air, respectively. Climate also goes through cycles that last much longer than a human lifetime. Natural climate records, such as ice cores, fossilized pollen, ocean sediments, and tree growth rings, provide a large portion of the information we have on previous climatic conditions.

Long- and short-term cycles exist in the climate of the world (Millar et al. 2006). The former can be directly linked to variations in the amount of solar energy that the earth receives because of the geometry of the Earth's orbit and its passage around the sun. Scientists have also suggested that variations in the tilt of the Earth's axis could be a contributing cause. This angle influences both the annual temperature range and the long-term effects on the Earth's climate. Last but not least, tectonic motion and ocean circulation are two elements that impact the planet's long-term climate. On the other hand, seasonal fluctuations in natural parameters, e.g., temperature, and air-sea interaction processes, such as monsoons and ENSO, are the most prevalent ways to identify short-term cycles.

Schlesinger and Ramankutty (1994) revealed the oscillation of the global climate over 65-70 years. This was further determined by Minobe (1997) for the sea surface temperature (SST) variations over the North Pacific. According to several hydrometeorological patterns of the global-scaled climate cycles, SST and pressure anomalies (Renard and Lall 2014), precipitation (Bhatia et al. 2020) vary on annual to multidecadal time scales. A coherent pattern of rhythmic fluctuations of roughly 60–90 years in the North Atlantic multidecadal SST has been recognized by Kerr (2000). Knudsen et al. (2011) linked a persistent 50-70 year Atlantic multidecadal SST oscillation to internal ocean-atmosphere variability. Maiyza et al. (2015) modelled the long-term trends of Monthly SST and sea surface salinity anomalies (MSSTA & MSSSA), for 63 years (1948-2010) within the Northern Atlantic and Pacific Oceans. Their results revealed that the MSSTA exhibited cyclic behaviour within the two basins, whereas the MSSSA showed this behaviour only within the Atlantic Ocean. They concluded that the MSSSA within the Pacific might reflect the absence of cyclic variations or a very large cycle of variations, a tiny part of it, appeared throughout the period of investigation. Maiyza et al. (2021) extended their work to examine the behaviour of the mean annual surface temperature and salinity anomalies (MASTA & MASSA) within the South Atlantic and the South Pacific Oceans. They used long-term data sets over 103 years (1911-2013) within the former, and 155 years (1859-2013) within the latter ocean. They concluded that the two oceans in the Southern Hemisphere exhibited periodic (cyclic) trends in both MASTA and MASSA. Their conclusions are in agreement with those of Venegas et al. (1998) who showed that the SST anomalies in the Southern Atlantic Ocean exhibit interdecadal cyclic variations.

Not only the physical properties, but also the fishing resources (fish catch) have been proven to have cyclic behaviour of abundance, e.g., 60-year cycle of Pacific Sardine abundance (Baumgartner et al. 1992), the 60-70 year oscillation of Icelandic Cod (Jonsson 1994) and 50-70 year cycles in the stock dynamics of the main commercial species in the Atlantic and Pacific Oceans (Klyashtorin 1998).

Cyclic Behaviour in the Southeastern Mediterranean Sea

Numerous studies were conducted in the southeastern Mediterranean Sea to determine whether variations in the hydrography and fisheries exhibit cyclic behaviour. Maiyza and Kamel (2009) investigated the SST trend of variations in the southeastern Mediterranean Sea over the period (1948-2008) and their findings support the idea that SST fluctuates over time rather than continuously rising because of the so-called global warming. The results of Maiyza et al. (2010) showed that the sea surface temperature anomaly (SSTA) in the southeastern Mediterranean basin had both positive and negative fluctuations over the period 1948-2008. These cycles' lengths varied between 8 and 15 years. Nearly all of these cycles coincide with the 11-year cycle of sunspot activity. This was also confirmed for the sea surface salinity anomaly (SSSA) over the period 1948-2010 (Maiyza and Kamel 2010). Examining fluctuations in air temperature and hydrography as well as the River Nile drought phenomenon reveals that the 70-year cycle may be one of the most significant drivers of climate change (Maiyza et al. 2011). The drought extends approximately +/- 7 years each time it occurs, according to all the references given in Table 1, with droughts of duration of less than five years not considered. The River Nile droughts are revealed to be more tied to the droughts in the entire Nile valley (regional) and not only in the Egyptian territory (local). It appears from Table 1 that the River Nile drought has a 70-year cycle that can be regarded as a periodic cycle, affecting it from BC up until the 1980s. The verified Nile drought cycle has a difference of 70 years, even across very long epochs.

Table 1. *Different Years of the River Nile Drought Based on Historical Data*

Difference (in years)	Year	Reference
2800=40*70	1590 B.C.	http://www.aawsat.com/leader.asp?section=3&article=537284&issueno=11258
490=7*70	1210	El-Fandy et al. (1994)
210=3*70	1700	El-Fandy et al. (1994)
70=1*70	1910	Flohn (1975) and El-Fandy et al. (1994)
	1980	El-Baz (1986 and 1989), El-Fandy et al. (1994), Gad El-Rab (1994) and Flohn (1975)

Source: Maiyza et al. (2011).

Moreover, according to the estimates of Maiyza et al. (2011) using the quadratic trend-line approach, it has been possible to predict that the warm phase of the climate cycle during the research time (1948-2010) would finish by 2016 and be replaced by a new cold phase. Verification of the veracity of these results and conclusions has begun to be felt by people through the severely cold winters hitting the Egyptian territories. The cyclic behaviour of variations of the wind pattern over the southeastern Mediterranean basin was examined by El-Geziry et al. (2013a) throughout the period 1956-1990. The cycle in the catch of different species matches the cycle in the behaviour of anomalies in air temperature, salinity, and SST that have been previously studied in the vicinity of the southeastern Mediterranean Sea (El-Geziry et al. 2013b). Their results also proposed the 70-year cycle to dominate the cycles of catch as in other regions worldwide. The general behaviour of the mean annual salinity anomaly (1948-2012) for the Intermediate and Deep Levantine waters exhibits the same tendency of cyclic changes deduced for the surface layer, although with varying magnitude and occurrence patterns (El-Geziry et al. 2019). This is explained by the fact that these two water masses share a Mediterranean wintertime origin (Wüst 1961). For the deep Levantine water, the trend is only fully realized quite close to the area where this water mass was formed (Wüst 1961, Maiyza 1984, El-Geziry et al. 2019). In the southerneastern Mediterranean region, the general behaviour of the mean annual temperature anomaly at all levels—subsurface, intermediate, and deep—showed the same trend for the surface layer, although with varying intensities and recurrence times (El-Geziry et al. 2021). The late winter vertical convection on the extremely cold and saltier water column with transparency (30 m) in this area can be responsible for this. Throughout the water column, at all examine levels, and even near the location where the deep Levantine water mass was formed, the trend of "cyclic" parabolic behaviour is evident.

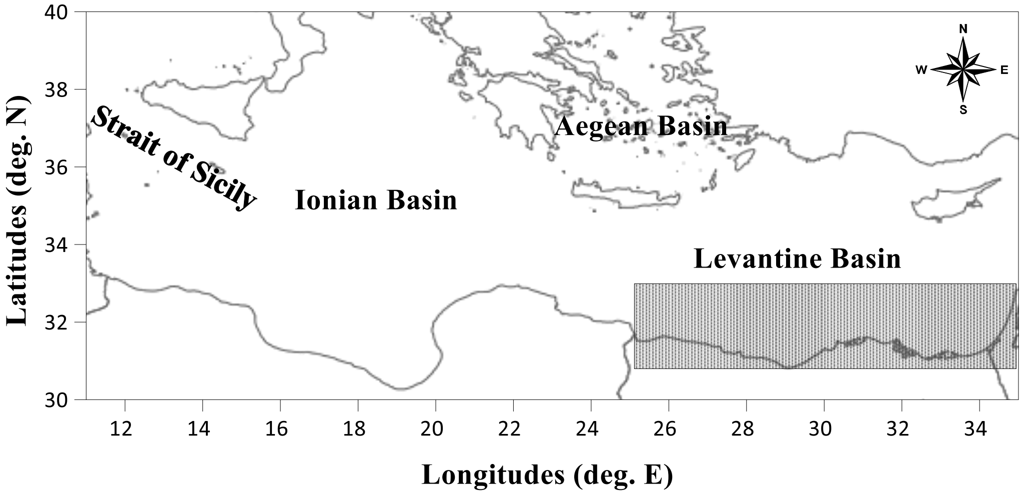
Ocean heat content is a crucial indicator of the shift in the Earth's radiative balance because 80% of the additional heat caused by anthropogenic warming now dwells in the ocean (Levitus et al. 2005). It also significantly contributes to the rise in sea level through thermal expansion (Domingues et al. 2008). Factually, Levitus et al. (2000) was the first to speak about the interannual-to-decadal variability of the ocean heat content. Since then, periodical updates and more investigations have been released. The observed temperature and salinity trends over the previous 30 years, as reported in Levitus et al. (2005), have substantially reversed since the mid-1990s in line with the North Atlantic Ocean (NAO) changing phase. This reflects a sign of a cyclic behaviour of variation. As it allows for the elimination of diurnal (full) and to a lesser extent monthly effects on the examined thermal behaviour, the heat content can be thought of as a more accurate component to measure the probable thermal cyclic behaviour within a specific basin.

The present work is on the same track of investigating the long-term cyclic behaviour of physical properties in the southeastern Mediterranean basin; considering the heat storage as a parameter of interest. The work aims to: (1) provide the best fit model of heat storage anomaly changes; and (2) investigate any cyclic behaviour of change over a considerable span.

Data and Methods of Analysis

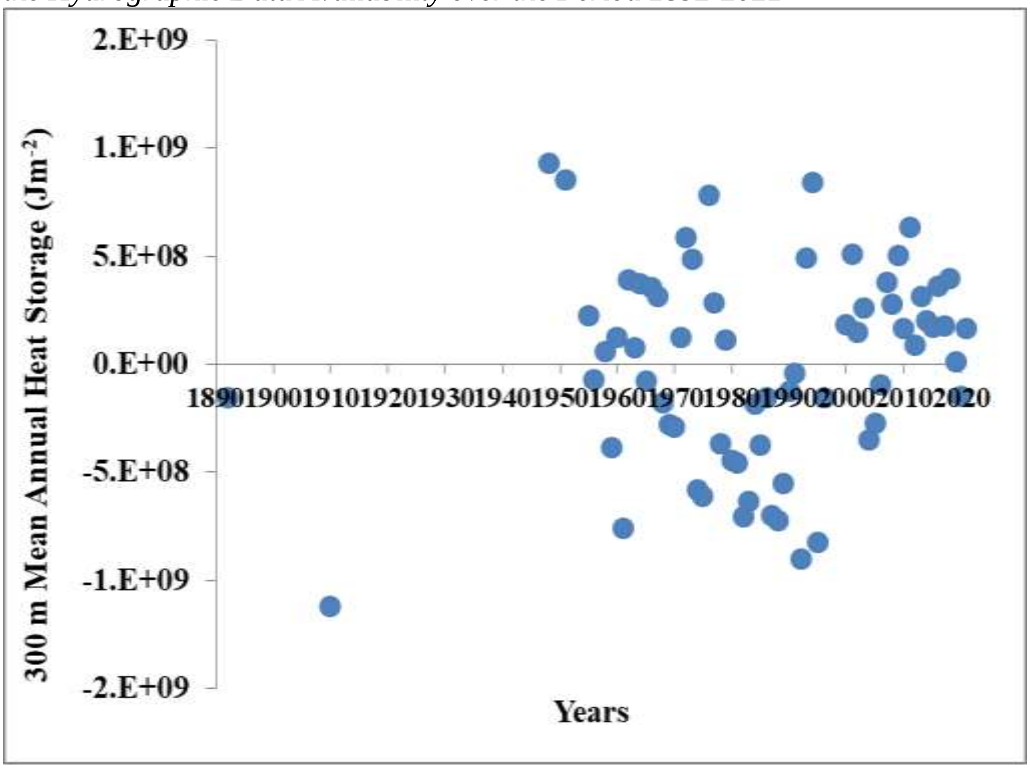
The current area of investigation is located in the southeasterrn Mediterranean Sea, southern Levantine border, off the coast of Egypt (Figure 1). Latitudinally, this region stretches from the Mediterranean coast of Egypt to 33° N, while longitudinally, it stretches from 25° E to the Asian shore.

Figure 1. *The Eastern Mediterranean Basin with Area of Investigation Shaded*



Source: Authors by using Surfer16® Software.

Figure 2. *The 300 m Mean Annual Heat Storage Anomaly Calculated Based on the Hydrographic Data Availability over the Period 1892-2021*



Source: Authors.

The hydrographic data (temperature and salinity) were scattered over the period 1889-2021 (Figure 2). The data files came from the Egyptian National Oceanographic Data Centre (ENODC), the World Data Centers (WDC) A and B (Washington and Moscow, respectively), and the Russian cruises through the Physical Oceanography of Eastern Mediterranean (POEM) project. However, because of the scarcity of data over the period 1892-1964, calculations of the mean annual heat storage anomaly in the present work were mainly based on the period 1965-2021 of continuous records.

The monthly heat storage was considered for the upper 300 m depth, known to comprise the surface, subsurface and upper level of the intermediate water masses in the eastern Mediterranean Basin (Maiyza 1993). The integrated monthly heat storage was calculated using the following equation:

$$h = \int_0^z \rho\, C_p\, T dZ \qquad (1)$$

Where,
h is the monthly heat storage (Jm^{-2})
ρ is the mean seawater density (kgm^{-3})
C_p is the mean specific heat capacity ($Jkg^{-1}C^{-1}$)
T is the mean seawater temperature (°C)

The specific heat capacity was calculated using the following equation (Korne 1972):

$$C_p = 4186\,[1.0049 - 0.001621\,S + (3.5261 \times 10^{-6}\,S^2) - \{(3.2506 - 0.1479\,S + 7.7765 \times 10^{-4} S^2)10^{-4} T\} + \{(3.8103 - 0.12084\,S + 6.121 \times 10^{-4} S^2)10^{-6} T^2\}] \qquad (2)$$

The integrated monthly heat storage anomaly was calculated as:

$$\Delta h = h - h_m \qquad (3)$$

Where,
Δh is the mean monthly heat storage anomaly (Jm^{-2})
h is the calculated monthly heat storage using Eq. (1)
h_m is the climatologic heat storage over a specific month (Jm^{-2}).

Using the entire set of available monthly event in every year over the period 1965-2021, the mean annual heat storage anomaly (ΔH; Jm^{-2}), which is the mean monthly heat storage over a specific year, was calculated, using the following equation:

$$\Delta H = \sum \frac{\Delta h_m}{n} \qquad (4)$$

Where n is the number of events

The best fitted representational models of the mean annual heat storage anomaly have been investigated using time series analysis using the ordinary least squares approach. Based on the highest determination coefficient (R^2) value and the lowest standard error of its estimation for the model, the represented models have been chosen. Using the SPSS® software, the significance of regression coefficients and the entire model were evaluated in accordance with the results of the t- and the F- tests. The models represent mean annual heat storage anomalies as dependent variables (Y_{300}) and the time (t) as an independent variable.

Results and Discussion

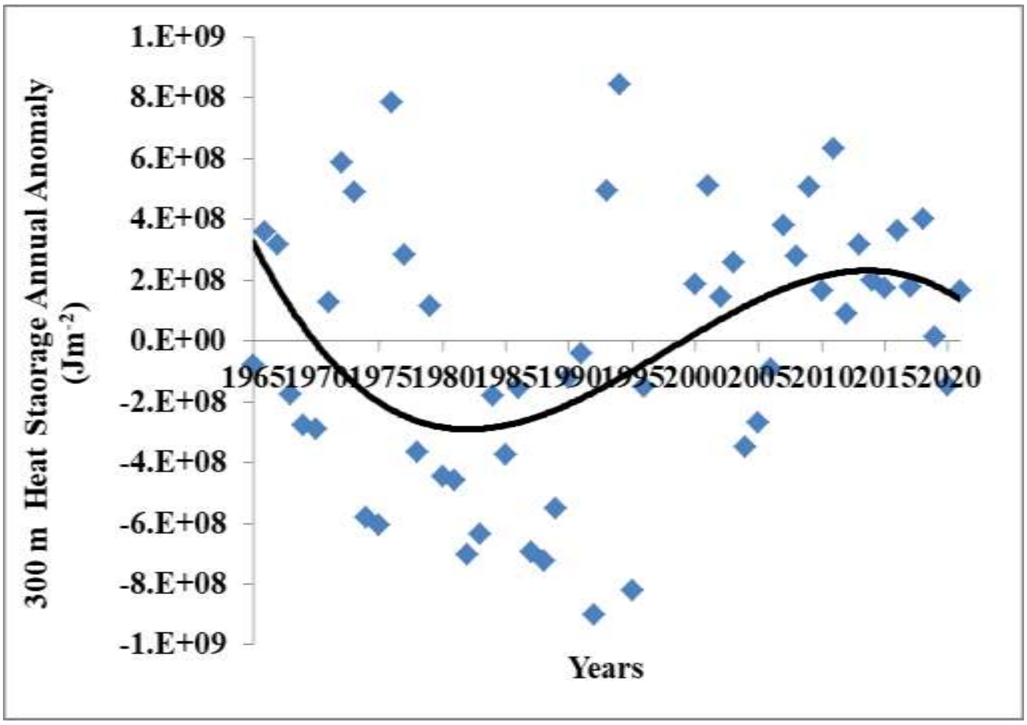
The minimum mean annual heat storage anomaly of the 300 m layer ranged from -9.0E9 Jm^{-2} (1992) to +0.84E9 Jm^{-2} (1994), with an overall mean of -0.016E9 Jm^{-2} during the period (1965-2021). +0.43E9 Jm^{-2} was the standard deviation from the mean. Figure 3 depicts the mean annual heat storage anomaly during the specified period (1965-2021). The following cubic equation provides the best fit model to represent variability in the present mean annual heat storage anomaly dataset:

$$Y_{300} = -38205.47\, t^3 + 3707343\, t^2 - 9E7\, t + 4.3E8$$

The R^2 coefficient of the model was 0.21, with a standard error of the estimates of +0.4E9 Jm^{-2}.

According to this best-fit model, the minimum mean annual heat storage anomaly occurred in 1980 and the maximum was in 2015. This reveals a 70-year cycle of variation in the heat storage of the upper 300 m in the southeastern Mediterranean basin. This result confirms the conclusion of Maiyza et al. (2011) that the anticipated warm part of climate cycle in the southeastern Mediterranean region would end by the 2016 to start a cold cycle part. It also coincides with the 70-year cycle in the southeastern Mediterranean catch concluded by El-Geziry et al. (2013b), which, itself, was in agreement with the cyclic behaviour concluded for the hydrographical (Maiyza and Kamel 2009, 2010, Maiyza et al. 2010) and air-temperature conditions (Maiyza et al. 2011) affecting the area of investigation. The present result comes also in agreement with the conclusion of Levitus et al. (2005) that the observed temperature and salinity trends have substantially reversed since the mid-1990s in line with the North Atlantic Ocean (NAO) changing phase. Iona et al. (2018) concluded a 40-year cycle of fluctuation in the heat content of the whole Mediterranean basin during the period 1950-2015; following the Atlantic Multidecadal Oscillation climate cycle. They distinguished three intervals of change in the Mediterranean heat content: a decrease from 1960 to 1980, followed by a stable content from 1980 to 1990, followed by an increase from 1990 to 2015.

Figure 3. *The 300 m Mean Annual Heat Storage Anomaly for the Specified Period 1965-2021 with the Best Fit Model Shown in Black Line*



Source: Authors.

Based on the present results, the perspective of future work is highly recommended to involve the examination of the relationship between the values of the inversion periods and the occurrence of extreme events of floods and heat waves, forest fires, drought and flooding of lakes and rivers.

Conclusion

To conclude, the 70-year cycle of variation in the hydrographic conditions in the southeastern Mediterranean region is confirmed in the present work, taking the heat storage parameter as examined parameter. Heat storage is said to be a more accurate component to measure the probable thermal cyclic behaviour within a specific basin; as it allows for the elimination of diurnal (full) and to a lesser extent monthly effects on the examined thermal behaviour.

References

Bačević N, Valjarević A, Kićović D, Milentijević N, Ivanović M, Mujević M (2020) Analysis of air temperature trends: city of Podgorica (Montenegro). *The University Thought - Publication in Natural Sciences* 10(1): 31–36.

Baumgartner TR, Soutar A, Ferreira-Bartrina V (1992) Reconstruction of the history of Pacific sardine and Northern Pacific anchovy populations over the past two millennia from sediments of the Santa Barbara basin, California. *California Cooperative Ocean Fisheries Investigations Report* 33: 24-40.

Bhatia N, Singh VP, Lee K (2020) Sensitivity of extreme precipitation in Texas to climatic cycles. *Theoretical and Applied Climatology* 140(3): 905–914.

Bonacci O, Bonacci D, Patekar M, Pola M (2021) Increasing trends in air and sea surface temperature in the Central Adriatic Sea (Croatia). *Journal of Marine Science and Engineering* 9(4): 358.

Domingues CM, Church JA, White NJ, Gleckler PJ, Wijffels SE, Barker PM, et al. (2008) Improved estimates of upper-ocean warming and multi-decadal sea-level rise. *Nature* 453(7198): 1090–1093.

El-Baz F (1986) Space photographs of drought in Africa. In *Lunar and Planetary Science XVII*, 200–201. Houston, Texas: Lunar and Planetary Institute.

El-Baz F (1989) Monitoring Lake Nasser by space photography. Remote Sensing and Large-Scale Global Processes. In *Proceedings of the IAHS 3rd International Assembly*, 177–181.

El-Fandy MG, Ashour ZH, Taiel SMM (1994) Time series models adoptable for forecasting Nile floods and Ethiopian rainfalls. *Bulletin of the American Meteorological Society* 75(1): 83–94.

El-Geziry TM (2021) Analysis of long-term rainfall and air temperature patterns in Alexandria (Egypt). *Arabian Journal of Geoscience*s 14(13): 1199.

El-Geziry T, Maiyza I, Maiyza H (2013a) General trend of wind anomalies in the South Eastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 24(2): 85–97.

El-Geziry T, Maiyza I, Abdel-Hafez S, Maiyza Sh, Kamel M (2013b) Interannual variability of the South-eastern Mediterranean catch and its relation to hydrographical and air-temperature anomalies. *Journal of King Abdulaziz University: Marine Sciences* 24(1): 43–54.

El-Geziry TM, Maiyza IA, Kamel MS (2019) Salinification in the Southeastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 29(2): 1–12.

El-Geziry TM, Maiyza IA, Kamel MS (2021). Behaviour of temperature variations in subsurface layers in the South-Eastern Mediterranean Sea. *Athens Journal of Sciences* 8(1): 9–22.

Flohn H (1975) History and intransitivity of climate. In *World Climate Data Programme, WMO, Geneva, WMO/GARP Publications Series* 16: 106–117.

Gad El-Rab R (1994) *Heat, salt and water budget of Lake Nasser*. MSc Thesis. Faculty of Science, Assuit University.

Ilyes C, Szűcs P, Turai E (2022) Appearance of climatic cycles and oscillations in Carpathian Basin precipitation data. *Hungarian Geographical Bulletin* 71(1): 21–37.

Iona A, Theodorou A, Sofianos S, Watelet S, Troupin C, Beckers J-M (2018) Mediterranean Sea climatic indices: monitoring long-term variability and climate changes. *Earth System Science Data* 10(4): 1829–1842.

IPCC (2013) *Climate change 2013: the physical science basis*. Cambridge, UK: Cambridge University Press.

Jonathan KH, Raju PS (2017) Analysis of rainfall pattern and temperature variations in three regions of Sultanate of Oman. *International Journal of Civil Engineering and Technology* 8(2): 173–181.

Jonsson J (1994) Fisheries off Iceland, 1600-1900. In *ICES Marine Science Symposium* 198: 3–16.

Kerr RA (2000) A North Atlantic climate pacemaker for the centuries. *Science* 288(5473): 1984–1985.

Klyashtorin LB (1998) Long-term climate change and main commercial fish production in the Atlantic and Pacific. *Fisheries Research* 37(1–3): 115–125.

Knudsen MF, Seidenkrantz M-S, Jacobsen BH, Kuijpers A (2011) Tracking the Atlantic multidecadal oscillation through the last 8,000 years. *Nature Communications* 2(1): 178.

Korne RA (1972) *Marine chemistry (water composition and chemistry of hydrosphere)*. Moscow: USSR (in Russian).

Levitus S, Antonov J, Boyer T (2005) Warming of the world ocean, 1955–2003. *Geophysical Research Letters* 32(2): L02604.

Levitus S, Antonov JI, Boyer TP, Stephens C (2000) Warming of the world ocean. *Science* 287(Mar): 2225–2229.

Maiyza IM (1984) *Long term variation of water temperature in the eastern part of the Mediterranean Sea*. PhD Thesis. Moscow University (in Russian).

Maiyza IM (1993) Heat storage in the Eastern Mediterranean. *Journal of Physical Oceanography* 23(6): 1259–1263.

Maiyza I, Kamel M (2009) Climatological trend of sea surface temperature anomalies in the South Eastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 20(1): 59–66.

Maiyza I, Kamel MS (2010) Climatological trend of sea surface salinity anomalies in the South Eastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 21(2): 63–72.

Maiyza IA, El-Geziry TM, Maiyza HI (2021) Trends of variations in the mean annual surface temperature and salinity anomalies in the Southern Hemisphere. *Indian Journal of Geo-Marine Science* 50(4): 271–276.

Maiyza I, Said M, Kamel M (2010) Sea surface temperature anomalies in the South Eastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 21: 151–159.

Maiyza I, El-Geziry T, Maiyza H, Kamel M (2011) Climatological trend of air temperature anomalies in the South Eastern Mediterranean Sea. *Journal of King Abdulaziz University: Marine Sciences* 22(2): 55–65.

Maiyza IA, El-Geziry TM, Maiyza HI, Kamel MS (2015) On the long-term variations of the surface hydrographic anomalies in the Northern Hemisphere. In *Proceedings of the 1st International Conference of Science, Engineering and Environment (SEE)*, 457–461.

Masson V, Zhai P, Pirani A, Connors SL, Pean C, Berger S, Caud N, et al. (2021) *IPCC, 2021: summary for policymakers, in: Climate Change 2021: The Physical Science Basis*. UK: Cambridge University Press.

Millar C, Neilson R, Bachelet D, Drapek R, Lenihan J (2006) *Climate change at multiple scales, in: forests, carbon and climate change*. USA: Oregon Forests Institute.

Minobe S (1997) A 50–70 year climatic oscillation over the North Pacific and North America. *Geophysical Research Letters* 24(6): 683–686.

Piticar A, Ristoiu D (2014) The influence of changes in teleconnection pattern trends on temperature and precipitation trends in northeastern Romania. *Riscuri Si Catastrofe* XIII: 109–122.

Renard B, Lall U (2014) Regional frequency analysis conditioned on large-scale atmospheric or oceanic fields. *Water Resource Research* 50(12): 9536–9554.

Schlesinger ME, Ramankutty N (1994) An oscillation in the global climate system of period 65–70 years. *Nature* 367(Feb): 723–726.

Thapa A, Silwal A, Gautam SP, Nepal CK, Bhattarai S, Timsina D (2021) Surface air temperature trends in Kathmandu Valley for 2011-2017. *BIBECHANA* 18(2): 95–104.

Tonbol KM, El-Geziry TM, ElBessa M (2018) Evaluation of changes and trends in air temperature within the Southern Levantine basin. *Weather* 73(2): 60–66.

Wüst G (1961) On the vertical circulation of the Mediterranean Sea. *Journal of Geophysical Research* 66(10): 3261–3271.

Venegas SA, Mysak LA, Straub DN (1998) An interdecadal climate cycle in the South Atlantic and its links to other ocean basins. *Journal of Geophysical Research* 103(C11): 24723–24736.

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Some New Properties of Cyclotomic Cliques Arrangements

By Philippe Ryckelynck[*] & Laurent Smoch[±]

In this paper, we study the arrangement of lines in the euclidean plane constructed from the geometric clique graph generated by the regular n -gon. The vertices of this clique arrangement are located on finite concentric circles that we call orbits. We focus especially on the number of orbits and the number of vertices inside and outside the regular n-gon. Combinatorics in finite sets of quadruplets of integers provide information on the way the orbits are distributed. Next, using cyclotomic fields, we give galoisian properties of the radii of the orbits and their cardinalities.

Keywords: *arrangement of lines in the plane, cyclotomic fields, geometric graphs, Galois theory*

Introduction, Notation and Historical Notice

Let n be an integer and let $\mathcal{C}_n$ be the regular n-gon with vertices $z_m = \exp\left(\frac{2Im\pi}{n}\right)$, where $0 \leq m < n$ and $I = \sqrt{-1}$. If i, j, k, ℓ are four conveniently chosen indices then the two straight-lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ defined respectively by the two couples (z_i, z_j) and (z_k, z_ℓ) are secant and we denote $z_{i,j,k,\ell}$ as their intersection point. Let $\mathcal{K}_n$ the *geometric graph* which consists of the union of $\mathcal{C}_n$ together with the sets of straight lines $\mathcal{D}_{i,j}$ and intersection points $z_{i,j,k,\ell}$. We partition $\mathcal{K}_n$ in three sets as follows $\mathcal{K}_n = \mathcal{C}_n \cup \mathcal{K}_{e,n} \cup \mathcal{K}_{i,n}$, by defining respectively $z \in \mathcal{K}_{e,n}$ and $\mathcal{K}_{i,n}$ if and only if $|z| > 1$ and $|z| < 1$. The arrangement of lines in the plane which consists of the various straight lines $\mathcal{D}_{i,j}$, passing through the points z_i and z_j for all indices $i \neq j$ may be referred as the *clique arrangement* constructed from $\mathcal{C}_n$.

The aim of this paper is to state that all the intersection points $z_{i,j,k,\ell}$ are located on *circular orbits* centered at the origin, but also to characterize the radii of these orbits and to describe accurately their distribution. We will denote by $M_{e,n}$ and $M_{i,n}$ the respective numbers of exterior and interior orbits of the arrangement $\mathcal{K}_n$. We set $M_n = 1 + M_{e,n} + M_{i,n}$ which stands for the total number of orbits. Likewise, we will denote by $N_{e,n}$ and $N_{i,n}$ the respective cardinalities of the two sets $\mathcal{K}_{e,n}$ and $\mathcal{K}_{i,n}$.

Let $N_n = n + N_{e,n} + N_{i,n}$ be the whole number of intersection points.

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Table 1. *Number of Intersection Points or Vertices of the clique* $\mathcal{K}_n$

n	3	4	5	6	7	8	9	10	11	12	13	14	...
N_n	3	5	15	37	91	145	333	471	891	901	1963	2185	…

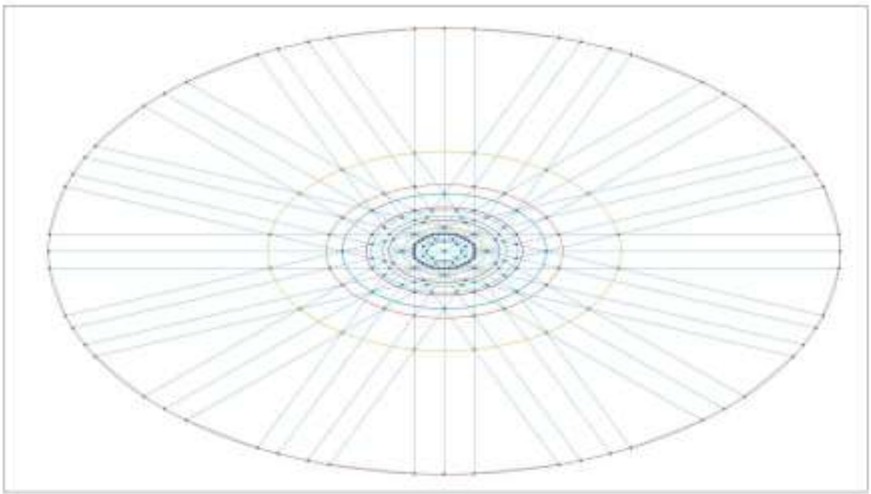
The sequence $(N_n)_{n\geq 3}$ is referred as A146212 in Sloane's (2023) OEIS. For odd n, it has been noted that $N_n = \frac{1}{8}n(n^3 - 7n^2 + 15n - 1)$ (see OEIS). We shall prove in Section 3 that these values N_n, which are not provided from a closed formula, may be however upper-bounded using inequalities on number lines by the sequence which starts as follows

Table 2. *Upper Bounds for Intersection Points of the clique* $\mathcal{K}_n$

n	3	4	5	6	7	8	9	10	11	12	13	14	...
q''_n	3	5	15	39	91	182	333	560	891	1347	1963	2765	…

Let us mention first that OEIS does not report this sequence, second that the upperbound is optimal for n even and at last, that the construction of the sequence (q''_n) is provided by a closed formula independent with respect to the parity of n.

Figure 1. *The Cyclotomic Clique Arrangement for* $n = 8$ *Together with its Circular Orbits*



Many researchers have been interested in describing and enumerating intersection points and polygons generated from an arrangement of lines and in particular from the diagonals of a regular n-gon. We may cite of course the results obtained by Bol (1936), Steinhaus (1958, 1983), Harborth (1969a, b), Tripp (1975), Rigby (1980) and especially those presented by Poonen and Rubinstein (1998), who were the first to provide two remarkable formulas for the number of *interior intersection points* made by the diagonals of a regular n-gon and the associated number of regions. Let us mention that the two authors provide also a brief but very interesting historical notice on the subject.

As an alternative of the formulas of Theorems 1 and 2 in Ryckelynck and Smoch (2023) related respectively to the number of interior points $N_{i,n}$ and the number of regions, one may devise two friendly written algorithms yielding those numbers. It is remarkable that the final outputs of these algorithms are polynomials on each residue class modulo 2520. Here are the first values of $N_{i,n}$

Table 3. *Sequence ($N_{i,n}$) A006561 in OEIS*

n	3	4	5	6	7	8	9	10	11	12	13	14	...
$N_{i,n}$	0	1	5	13	35	49	126	161	330	301	715	757	…

By substracting these values to the total number points of the graph and the points on the unit circle, i.e., by calculating $N_{e,n} = N_n - n - N_{i,n}$, we get the sequence of exterior points numbers of the cliques arrangements

Table 4. *Sequence ($N_{e,n}$) A146213 in OEIS*

n	3	4	5	6	7	8	9	10	11	12	13	14	...
$N_{e,n}$	0	0	5	18	49	88	198	300	550	588	1235	1414	…

This sequence is defined, for n odd given only, through the formula $N_{e,n} = n(2n^3 - 15^2 + 34n - 21)/24$. The authors also compute the number of regions formed by the diagonals, by using Euler's formula $V - E + F = 2$.

In the recent paper (Ryckelynck and Smoch 2023), we got interest in cyclotomic arrangements of lines in the plane. We initiated a new approach to the problem of enumerating convex connected components (known as chambers), depending on their shapes and their compacity. We presented, in an analogous problem, the description of each chamber of the *boundification* of the space defined as the complement of an arrangement of lines associated to the regular n-gon. Symmetry, circular orbits, goniometric functions are the core of this paper. Nevertheless, the whole program of work given in this work was too much difficult to replicate in the case of the clique arrangement $\mathcal{K}_n$ of the present paper, and we take as an objective to explain the way the intersection points are in fact "regularly" distributed along some concentric circles.

Let us present the notation used throughout this paper. First, $\sharp(S)$ denotes the cardinality of some finite set S, $\Im(f)$ denotes the range of the mapping $f: S_1 \to S_2$. As usual, if $x \in \mathbb{R}$ then $[x]$ denotes the greatest integer function. Let us use the abbreviation $pq_{\pm} = (p \pm q)\frac{\pi}{n}$ for convenient integer indices p and q. Although we may use indices modulo n, we impose instead the inequalities $0 \leq i \leq n-1$ over all subsequent indices.

Next, $\varphi(n)$ denotes as usual the Euler's totient function. Let $\zeta_n = \exp\left(\frac{2\pi}{n}I\right)$ be a n-th primitive root of unity so that $\zeta_n^n = 1$, and let $\mathbb{Q}(\zeta_n)$ be the cyclotomic field generated by ζ_n.

For various integers n, i, j, k, ℓ, such that $i + j - (k + \ell)$ is not a multiple of n we set

$$J_n(i,j,k,\ell) = \frac{\cos^2(ij_-) + \cos^2(k\ell_-) - 2\cos(ij_-)\cos(k\ell_-)\cos(ij_+ - k\ell_+)}{\sin^2(ij_+ - k\ell_+)}.$$

When $k + \ell \not\equiv 0 \bmod n$ and $i + j \equiv 0 \bmod n$, we set

$$\tilde{J}_n(i,j,k,\ell) = \frac{\cos^2(ii_+) + \cos^2(k\ell_-) - 2\cos(ii_+)\cos(k\ell_-)\cos(k\ell_+)}{\sin^2(k\ell_+)}.$$

Let us consider the following finite set $\mathcal{R}_n = \Im(J_n) \cup \Im(\tilde{J}_n)$. The numbers in $\mathcal{R}_n$ are nothing but the squares of the radii of the orbits containing the vertices of $\mathcal{K}_n$. We have $\mathcal{R}_n \subset \left[0, \left(\sin\left(\frac{\pi}{n}\right)\right)^{-2}\right]$. Defining $\mathcal{R}_n$ shall allow to characterize the numbers $M_{i,n}$ and $M_{e,n}$ since

$$M_{i,n} = \sharp(\mathcal{R}_n \cap [0,1[) \text{ and } M_{e,n} = \sharp\left(\mathcal{R}_n \cap]1, \left(\sin\left(\frac{\pi}{n}\right)\right)^{-2}]\right).$$

The rest of this paper is organized as follows. Next section presents some properties of the geometric graph associated to the clique while the section after focuses mainly on the mappings J_n and $\tilde{J}_n$ which give the radii of the circles on which lie the intersection points of the graph $\mathcal{K}_n$. The next section is devoted to combinatorics inside the domain of J_n consisting in quadruplets (i,j,k,ℓ) of integers constrained by some specific inequalities. Afterwards, we use galoisian properties of $\mathbb{Q}(\zeta_{2n})$ in order to highlight the number of orbits associated to $\mathcal{K}_n$ and the numbers of vertices lying on those orbits. In the end, we conclude with open problems.

The Geometric Graph Associated to the Clique Arrangement

In this section, we determine the coordinates of the intersection points of the clique arrangement $\mathcal{K}_n$ as well as the modulii of these points. Next, we deal with symmetries and multiplicities. To define properly the vertices $z_{i,j,k,\ell}$, we may assume that $0 \le i < j \le n-1,\ 0 \le k < \ell \le n-1,\ i \le k$.

Lemma 1. We have $\{i,j\} \cap \{k,\ell\} \neq \emptyset \Leftrightarrow z_{i,j,k,\ell} \in \mathcal{C}_n$.

Proof. The implication $(\Rightarrow)$ is obvious. The converse implication relies on the following property resulting from a convexity argument: no three distinct vertices z_i, z_j and z_k of the regular n-gon are aligned. □

The intersection points $z_{i,j,k,\ell}$ are divided into two classes, depending on whether or not the indices $i + j$ and $k + \ell$ are equal to n or equivalently whether or not

$\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are vertical straight lines. Let us mention that when the integers i, j, k, ℓ are chosen in such a way that $i + j = k + \ell = n$ then the two straight-lines (z_i, z_j) and (z_k, z_ℓ) are parallel and disjoint and consequeltly, no intersection point occurs. The first class of intersection points is described as follows.

Lemma 2. We suppose that $i + j, k + \ell, n$ are three distinct integers. The parallax of the angular segment in the unit disk limited by O and the vertices $z_{i,j}$ and $z_{k,\ell}$, seen from $z_{i,j,k,\ell}$, is equal to $\theta = ij_+ - k\ell_+ = \big(i + j - (k + \ell)\big)\frac{\pi}{n}$. The cartesian coordinates of the intersection point $z_{i,j,k,\ell} = \mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}$ are equal to

$$x_{i,j,k,\ell} = \frac{\sin(ij_+)\cos(k\ell_-)-\cos(ij_-)\sin(k\ell_+)}{\sin(ij_+)\cos(k\ell_+)-\cos(ij_+)\sin(k\ell_+)} = \frac{\sin(ij_+)\cos(k\ell_-)-\cos(ij_-)\sin(k\ell_+)}{\sin(\theta)},$$

$$y_{i,j,k,\ell} = \frac{\cos(ij_-)\cos(k\ell_+)-\cos(ij_+)\cos(k\ell_-)}{\sin(ij_+)\cos(k\ell_+)-\cos(ij_+)\sin(k\ell_+)} = \frac{\cos(ij_-)\cos(k\ell_+)-\cos(ij_+)\cos(k\ell_-)}{\sin(\theta)},$$

while $|z_{i,j,k,\ell}|^2 = J_n(i,j,k,\ell)$.

Proof. Straightforward computations provide the equation of the straight line $\mathcal{D}_{i,j}$ passing through z_i and z_j ($i \neq j$ and $i + j \neq n$)

$$\mathcal{D}_{i,j}: y = -\cot(ij_+)x + \frac{\cos(ij_-)}{\sin(ij_+)}.$$

Solving the system of equations for $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ yields the cartesian coordinates of $z_{i,j,k,\ell}$. Squaring and adding these coordinates gives the formula for the square of the modulus. □

The second class of intersection points consists in the points $z_{i,j,k,\ell}$ where $i + j$ or $k + \ell$ is equal to n.

Lemma 3. Let us choose integers i, j, k, ℓ such that $i + j = n$ and $k + \ell \neq n$. Then the cartesian coordinates of the intersection point $\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}$ are equal to

$$x_{i,j,k,\ell} = \cos(ii_+), \qquad y_{i,j,k,\ell} = \frac{\cos(k\ell_-) - \cos(k\ell_+)\cos(ii_+)}{\sin(k\ell_+)},$$

while $|z_{i,j,k,\ell}|^2 = \tilde{J}_n(i,j,k,\ell)$. In the case where the integers i, j, k, ℓ are such that $i + j \neq n$ and $k + \ell = n$, $x_{i,j,k,\ell}$, $y_{i,j,k,\ell}$ and $|z_{i,j,k,\ell}|$ are obtained by replacing the quadruplet (i, j, k, ℓ) by (k, ℓ, i, j).

The proof is similar to the previous one by considering that this time $\mathcal{D}_{i,j}$ (or $\mathcal{D}_{k,\ell}$) is a vertical straight line.

Let us recall by the way that, given two straight lines with slopes s_1, s_2, the angle θ between these lines verifies the relationship $\tan\theta = \left|\frac{s_1 - s_2}{1 + s_1 s_2}\right|$. The slopes of $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are given by $s_1 = -\cot(ij_+)$ and $s_2 = -\cot(k\ell_+)$, and we have

$$\tan\theta = \left|\frac{\cot(ij_+) - \cot(k\ell_+)}{1 + \cot(ij_+)\cot(k\ell_+)}\right| = \left|\big(\cot(k\ell_+ - ij_+)\big)^{-1}\right|,$$

using the addition formula for cotangent. Hence, $\theta = \frac{\pi}{n}|k+\ell-i-j|$. This angle is the analog of the *parallax* in astronomy since it subtends any chord from z_i or z_j to z_k or z_ℓ. □

The upper bound for the number of intersection points N_n given in the introduction is nothing else than

$N_n \leq q_n'' =$

$n + \sharp\{(i,j,k,\ell), i<j, k<\ell, i<k, j\neq \ell, j\neq k, i+j-(k+\ell) \not\equiv 0 \bmod n\}.$

These various inequalities and constraints forbid the intersection points to be over-counted.

Let us introduce the rotation ρ with center at origin and angle $\frac{2\pi}{n}$ defined as $\rho(z) = z\exp\left(\frac{2I\pi}{n}\right)$ in the plane $\mathbb{C} \simeq \mathbb{R}^2$. Obviously, one has $\rho^n = Id$,

$\rho(z_p) = z_{p+1}$ and $\rho^m(z_p) = z_{p+m}$ for all $m \in \mathbb{N}$ and all $p \in \{0, \dots, n-1\}$.

We see also that $\rho(\mathcal{D}_{i,j}) = \mathcal{D}_{i+1,j+1}$ and that the vertex $z_{i,j,k,\ell}$ transforms as follows

$\rho(z_{i,j,k,\ell}) = \rho(\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}) = \mathcal{D}_{i+1,j+1} \cap \mathcal{D}_{k+1,\ell+1} = z_{i+1,j+1,k+1,\ell+1}.$

As a consequence,

$$\begin{aligned} x_{i+1,j+1,k+1,\ell+1} &= x_{i,j,k,\ell}\cos\left(\frac{2\pi}{n}\right) - y_{i,j,k,\ell}\sin\left(\frac{2\pi}{n}\right), \\ y_{i+1,j+1,k+1,\ell+1} &= x_{i,j,k,\ell}\sin\left(\frac{2\pi}{n}\right) - y_{i,j,k,\ell}\cos\left(\frac{2\pi}{n}\right). \end{aligned}$$

More generally, $\rho^m(z_{i,j,k,\ell}) = z_{i+m,j+m,k+m,,\ell+m}$ for all $m \in \mathbb{N}$.

In addition, symmetry considerations provide the following results

$z_{i,j,k,\ell} = z_{k,\ell,i,j}, \qquad z_{i+n,j+n,k,\ell} = z_{i,j,k+n,\ell+n} = z_{i,j,k,\ell}.$

The central symmetry holds only for even n and is contained in the following formula.

$-z_{i,j,k,\ell} = z_{i+\frac{n}{2},j+\frac{n}{2},k,\ell} = z_{i,j,k+\frac{n}{2},\ell+\frac{n}{2}}.$

So, according to rotations, we see that each orbit contains a multiple of n points. Let us arrange the orbits $\omega_1, \dots, \omega_{M_n}$ of $\mathcal{K}_n$ by increasing values of radius. For each p, the number of vertices of ω_p is a multiple $\nu_{n,p}n$ of n. So we get

$$N_{2n} = 1 + n\sum_{p=1}^{M_{2n}-1} \nu_{2n,p} \geq 1 + n(M_{2n}-1),$$

$$N_{2n+1} = n\sum_{p=1}^{M_{2n+1}} \nu_{2n+1,p} \geq nM_{2n+1}.$$

We give below, for $n = 8, 9, 10$, for the sake of conciseness, the cardinalities of the orbits of $\mathcal{K}_n$. To be cautious about the parity of n, we introduce the sequence Γ_n of lengths $M_{i,n} + 1 + M_{e,n}$ as follows

$$\Gamma_n = \begin{cases} \left(1, (nv_{n,p})_{p\in\{2,\dots,M_n\}}\right) & \text{for } n \text{ even} \\ (nv_{n,p})_{p\in\{1,\dots,M_n\}} & \text{for } n \text{ odd} \end{cases}.$$

In order to locate the n-gon $\mathcal{C}_n$, we underline its cardinality.

- Γ_8 =(1,8,8,8,16,8,$\underline{8}$,8,16,16,8,8,16,16),
- Γ_9 =(9,9,9,18,9,18,9,18,18,9,$\underline{9}$,9,9,18,9,18,18,9,18,18,18,9,18,9,9),
- Γ_{10} = (1,10,10,10,10,20,10,20,10,10,20,20,10, $\underline{10}$,10,20,20,10,20,20,10,20,10,20,20,20,10,20,10,20,20,20).

Proposition 1. For any integer $m \geq 1$, the sequences $(M_{mn})_{n\geq1}, (M_{e,mn})_{n\geq1}, (M_{i,mn})_{n\geq1}, (N_{mn})_{n\geq1}, (N_{e,mn})_{n\geq1}, (N_{i,mn})_{n\geq1}$ are increasing.

Proof. Indeed, for any integer m, if $z_{i,j,k,\ell}$ is a vertex of $\mathcal{K}_n$ then this point coincides with the vertex $z_{mi,mj,mk,m\ell}$ of $\mathcal{K}_{mn}$. Moreover, the inclusion keeps the various properties of "being inside" or "being outside" the unit-circle. Similarly, each straight line $\mathcal{D}_{i,j}$ of $\mathcal{K}_n$ coincides with the straight line $\mathcal{D}_{mi,mj}$ of $\mathcal{K}_{mn}$. So the sets of vertices and of edges of $\mathcal{K}_n$ are included in the respective sets of vertices and of edges of $\mathcal{K}_{mn}$. □

In the following table we give for $3 \leq n \leq 10$, the values of cardinalities $N_n, N_{e,n}, N_{i,n}$, and the numbers of various orbits $M_n, M_{e,n}, M_{i,n}$.

Table 5. *Numbers of Intersection Points and Orbits*

n	N_n	$N_{e,n}$	$N_{i,n}$	M_n	$M_{e,n}$	$M_{i,n}$
3	3	0	0	1	0	0
4	5	0	1	2	0	1
5	15	5	5	3	1	1
6	37	18	13	6	2	3
7	91	49	35	10	5	4
8	145	88	49	14	7	6
9	333	198	126	25	14	10
10	471	300	161	32	18	13

Table 5 illustrates both Proposition 1 and previous formulas.

Elementary Properties of the Radius Mappings $J_n(i,j,k,\ell)$ and $\tilde{J}_n(i,j,k,\ell)$

In this section, i, j, k, ℓ denote indices lying in $[0, n-1]$ and such that $\sharp\{i,j,k,\ell\} \geq 3$. We have two formulas expressing symmetry by change of pairs of lines, $J_n(k,\ell,i,j) = J_n(i,j,k,\ell)$, $\tilde{J}_n(k,\ell,i,j) = \tilde{J}_n(i,j,k,\ell)$. We notice that, for any index m, we have the obvious relations
$J_n(i,j,k,\ell) = J_n(i+m, j+m, k+m, \ell+m)$,
and

$\tilde{J}_n(i,j,k,\ell) = \tilde{J}_n(i+m, j+m, k+m, \ell+m)$.

These relations mean that the functions J_n and $\tilde{J}_n$ are in a sense invariant by rotations. So one may assume $i = 0$ without loss of generality when studying those mappings. Using Lemmas 1 to 3, we get the identity

$J_n(i+n, j+n, k, \ell) = J_n(i,j,k+n,\ell+n) = J_n(i,j,k,\ell)$.

The following property results from the examination of the explicit formulas given in previous section.

Lemma 4. Let $\{i,j,k,l\}$ such that the vertex $z_{i,j,k,\ell}$ exists. Then the four numbers $x_{i,j,k,\ell}$, $y_{i,j,k,\ell}$, $z_{i,j,k,\ell} = x_{i,j,k,\ell} + I y_{i,j,k,\ell}$, $|z_{i,j,k,\ell}|^2 = J_n(i,j,k,\ell)$ belong to the cyclotomic field $\mathbb{Q}(\zeta_{2n})$.

We emphasize on the fact that all the points z_m of the starting regular n-gon are in the cyclotomic field $\mathbb{Q}(\zeta_n)$, but however, the intersection points $z_{i,j,k,\ell} \notin \mathcal{C}_n$ lie in a larger field $\mathbb{Q}(\zeta_{2n})$.

When this is possible, in compliance with Galois's theorem, we may give explicit algebraic values of squared radii of orbits $\Xi_n = \left\{|z_{i+n,j+n,k,\ell}|^2\right\}$, which is done hereafter. Although Ξ_n being sets, we have sorted the values by increasing order. One has for instance $\Xi_3 = \{1\}$, $\Xi_5 = \left\{\frac{7-\sqrt{5}}{2}, 1, \frac{7+3\sqrt{5}}{2}\right\}$, $\Xi_6 = \left\{0, \frac{1}{3}, \frac{1}{2}, 1,3,7\right\}$ and by using old-school algebra, we see that Ξ_{10} contains

$$\{0,1,4\sqrt{5}+11,5+2\sqrt{5}, \frac{1}{2}(7+3\sqrt{5}), \frac{1}{2}(7-3\sqrt{5}),$$
$$\frac{1}{2}(7+\sqrt{5}), 4-\sqrt{5}, 4+\sqrt{5}, \frac{1}{2}(3+\sqrt{5})\}.$$

For the sake of conciseness, the list has been volontarily shorten up but all the values do have explicit expressions. The method of Archimedes allows to compute explicitly and algebraically each radii of each set Ξ_{2^n} starting with $\Xi_4 = \{0,1\}$. For example Ξ_8 is equal to

$$\{0,3-2\sqrt{2}, 1-\frac{1}{2}\sqrt{2}, \frac{1}{2}, 9-6\sqrt{2}, 1, 1+\frac{1}{2}\sqrt{2},$$
$$5-2\sqrt{2}, 3, 2+\sqrt{2}, 3+2\sqrt{2}, 5+2\sqrt{2}, 9+6\sqrt{2}\}.$$

But the formulas, although being known explicitly, become of a large complexity mainly due to Viète's formula.

Lemma 5. Let $w \in \mathbb{Q}(\zeta_N)$ be of modulus one, then $w = \zeta_N^m$ for some integer m, and conversely.

See the book by Washington (1997) for the proof. □

Lemma 6. We have $\sharp\{i,j,k,\ell\} = 3$ if and only if $J_n(i,j,k,\ell) = 1$ or $\tilde{J}_n(i,j,k,\ell) = 1$.

Proof. If a quadruplet is such that $\sharp\{i,j,k,\ell\} = 3$ with $i \neq j$ and $k \neq \ell$, it must be of the shape (i,j,i,ℓ) or (i,j,k,i) or (i,j,k,j) or (i,j,j,ℓ). Without loss of generality, we may suppose that $i = 0$ and that $k\ell = 0$. Up to a change of

notation, the value of the function J_n for all four preceeding quadruplets is equal to 1 in view of the equation
$\cos(x)^2 + \cos(y)^2 - 2\cos(x)\cos(y)\cos(x-y) = (\sin(x-y))^2$.
Let us suppose that $\sharp\{i,j,k,\ell\} = 4$. We show that the requirement J_n (or $\tilde{J}_n$) is equal to 1 is contradictory. Indeed, by using Lemma 4, if J_n (or $\tilde{J}_n$) is equal to 1, the point $z_{i,j,k,\ell}$ in $\mathbb{Q}(\zeta_{2n})$ has modulus equal to 1, so it is a root of unity, i.e. $z_{i,j,k,\ell} = z_m$ for some index $m \in \{0, \dots, n-1\}$. So the index m is repeated twice which induces the contradiction. □

From now on in this section, we assume that $\sharp\{i,j,k,\ell\} = 4$.

Proposition 2. Let us suppose that $j - i = \ell - k$, difference that we denote t_1 and let us denote $t_2 = k + \ell - i - j$. Then

$$J_n(i,j,k,\ell) = \frac{1}{2}\left(\frac{\cos\left(\frac{t_1\pi}{n}\right)}{\cos\left(\frac{t_2\pi}{2n}\right)}\right)^2.$$

If η is such a value, so is $\frac{1}{4\eta}$. Lastly, the number of these values is $(n-1)(n-2)$.

Proof. The explicit computation of J_n relies on the duplication formulas for sine and cosine. Since $\sharp\{i,j,k,\ell\} = 4$, the equality $t_1 = \frac{1}{2}t_2$ does not occur; indeed, otherwise $j = k = i + t_1$. So the value of J_n may not be equal to $\frac{1}{2}$.

Let $\eta = J_n(i,j,k,\ell)$ with $j - i = \ell - k = t_1$. Since we have $t_2 = 2(k-i)$, t_2 must be even. Now,

$$\frac{1}{4\eta} = \frac{1}{2}\left(\frac{\cos\left(\frac{t_2\pi}{2n}\right)}{\cos\left(\frac{t_1\pi}{n}\right)}\right)^2 = J_n(i',j',k',\ell'),$$

where the indices i', j', k', ℓ' are chosen so that

$$j' - i' = \ell' - k' = \frac{t_2}{2}, \quad k' + \ell' - i' - j' = 2t_1.$$

We get $j' = i' + \frac{t_2}{2}$, $k' = i' + t_1$, $\ell' = i' + t_1 + \frac{t_2}{2}$. These integers are all four distinct and satisfy the hypothesis of the lemma, i.e. $j' - i' = \ell' - k'$. Lastly, the number of those values is equal to $(n-1)(n-2)$ since, we choose arbitrarily the integer $\frac{t_2}{2}$ in $[1, n-1]$ and next an integer $t_1 \in [1, n-1]$ distinct of $\frac{t_2}{2}$. □

The Combinatorics of Quadruplets of Indices of Intersection Points

We denote $Q_n \subset [0, n-1]^4$ the set of quadruplets of integers (i,j,k,ℓ), $0 \le i,j,k,\ell \le n-1$ with $i \ne j$, $k \ne \ell$. Hence Q_n is in bijection with the set of pairs of well-defined lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ being parallel or concurrent through some vertex of $\mathcal{K}_n$. The commutative group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts transitively on Q_n as follows: if (g, b) is in this group, then

$$b \cdot (i,j,k,\ell) = (k,\ell,i,j) \text{ and}$$
$$g \cdot (i,j,k,\ell) \equiv (i+g, j+g, k+g, \ell+g) \bmod n.$$

The group $\mathbb{Z}/2\mathbb{Z}$ acts as the identity on the set of vertices of the arrangement while $\mathbb{Z}/n\mathbb{Z}$ acts as the rotation group on this set of vertices.

Lemma 7. The direct image of $\tilde{J}_n$ is included in the direct image of J_n while being distinct.

Proof. Let $\eta \in \Im(\tilde{J}_n)$ then there exist i,j,k,ℓ with $j = n-i$ such that $\eta = \tilde{J}_n(i,j,k,\ell) = |z_{i,j,k,\ell}|^2$. We choose any integer p such that $2p \equiv 0 \bmod n$ and $2p + k + \ell \not\equiv 0 \bmod n$. In view of the properties of ρ, we have also $\eta = |z_{i+p,j+p,k+p,\ell+p}|^2 = J_n(i+p, j+p, k+p, \ell+p)$, the indices characterizing the quadruplet avoiding the case of vertical straight lines as explained in a previous section. So $\eta \in \Im(J_n)$.

Conversely, let $\eta \in \Im(J_n)$ and let us show that $\eta \in \Im(\tilde{J}_n)$ except in the case when n is even and $i+j$ and $k+\ell$ are odd.

In the three opposite cases, we may choose an integer m so that the quadruplet $(i+m, j+m, k+m, \ell+m)$ satisfies either $i+j+2m \equiv 0 \bmod n$ or $k+\ell+2m \equiv 0 \bmod n$. Indeed, $\eta = J_n(i,j,k,\ell)$ with $i+j \neq 0 \bmod n$ and $k+\ell \neq 0 \bmod n$. We choose $m = \frac{1}{2}(n-i-j)$ or $m = \frac{1}{2}(n-k-\ell)$ and we have $\eta = J_n(i+m, j+m, k+m, \ell+m) = \tilde{J}_n(i+m, j+m, k+m, \ell+m)$ with at least one of the two conditions $2m \neq n-k-\ell$ or $2m \neq n-i-j$. Let us consider now the fourth remaining case, that is to say the case when n is even and $i+j$ and $k+\ell$ are odd. No transformation of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, except the identity, preserves the value $J_n(i+p, j+p, k+p, \ell+p)$ and accordingly the radius $J_n(i+p, j+p, k+p, \ell+p)$ cannot be sent to a radius of the shape $\tilde{J}_n(i', j', k', \ell')$. □

We may synthetize the proof by saying that, without changing the value of $J_n(i,j,k,\ell)$ or $\tilde{J}_n(i,j,k,\ell)$, one may apply a transformation $(i,j,k,\ell) \to (i+p, j+p, k+p, \ell+p)$, so either to remove the condition $i+j \equiv 0 \bmod n$ or to impose it.

Let $Q_n^{(3)} \subset Q_n$ denote the subset of quadruplets such that $\#\{i,j,k,\ell\} = 3$. Orbits of Q_n are either contained in $Q_n^{(3)}$ or in $Q_n - Q_n^{(3)}$. In each orbit contained in $Q_n - Q_n^{(3)}$, one may find a representative such that $i < j$ and $k < \ell$ and $i < k$ and $j \neq \ell$. This allows us to define the set Q'_n of quadruplets (i,j,k,ℓ) verifying

$$0 \leq i < j \leq n-1, \;\; 0 \leq k < \ell \leq n-1, \;\; i < k, \;\; j \neq \ell.$$

Now, the orbits contained in $Q_n - Q_n^{(3)}$ are identified with the set of admissible intersection points $z_{i,j,k,\ell} \notin \mathcal{C}_n$, and of points that do not exist due to parallelism,

since some of these quadruplets fail to define properly any intersection point because the underlying straight-lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ are parallel.
Let us denote Q_n'' the subset of Q_n' of which the elements are $(i,j,k,\ell) \in Q_n'$ satisfying $i + j \neq k + \ell \bmod n$. This implies that $z_{i,j,k,\ell}$ exists, since in that case the parallax $|i + j - k - \ell|\frac{\pi}{n}$ is not flat nor equal to zero. When $(i,j,k,\ell) \in Q_n'$ the value of $|i + j - k - \ell|$ is less than $2n - 2$, and thus the condition $i + j \neq k + \ell \bmod n$ is equivalent to $k + \ell - i - j \notin \{-n, 0, n\}$.
We define $q_n' = n + \#(Q_n')$, and $q_n'' = n + \#(Q_n'')$, both summands n counting the vertices of $\mathcal{C}_n$. In this section we will give explicit formulas for these two integer sequences. Note that the conditions in Q_n' are not necessary nor sufficient to define properly $z_{i,j,k,\ell}$, because the action of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ does not preserve these conditions, although it leaves invariant intersection points.
However, the conditions in Q_n' allow a crude estimation of the sequence (N_n) since we have:

Proposition 3. One has $\#(Q_n') = \frac{1}{8}n(n^3 - 6n^2 + 11n + 2)$,

$\#(Q_n'') = \frac{1}{16}n(2n^3 - 14n^2 - (-1)^n + 30n - 3)$, and $N_n \leq n + \sharp(Q_n'')$.

Proof. We first compute the cardinality of the set Q_n'. To do this, we split the cardinality as a sum of two triple sums of characteristic functions as follows

$$\#(Q_n') = \left[\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=i+1}^{j-1} + \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n-1}\right]\#\{\ell, \ell \neq j, \ell > k\} = s_1 + s_2,$$

the triple sum being distributed over the characteristic functions and where we have set

$$s_1 = \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=i+1}^{j-1}(n - k - 2)\,, s_2 = \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n-1}(n - k - 1).$$

Internal sums are easily computed:

$\sum_{k=i+1}^{j-1}(n - k - 2) = \frac{1}{2}(2nj - j^2 - 3j - 2ni - 2n + i^2 + 5i + 4)$ and
$\sum_{k=j+1}^{n-1}(n - k - 1) = \frac{1}{2}(n^2 - 3n - 2nj + j^2 + 3j + 2)$.

So we obtain

$$s_1 = \frac{1}{2}\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}(2nj - j^2 - 3j - 2ni - 2n + i^2 + 5i + 4),$$

$$s_2 = \frac{1}{2}\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}(n^2 - 3n - 2nj + j^2 + 3j + 2).$$

Once again we compute internal sums, say s_1' and s_2',

$s_1' = \frac{1}{2}\sum_{j=i+1}^{n-1}(j^2 - 3j - 2ij + i^2 + 3i + 2)$,
$s_2' = \frac{1}{2}\sum_{j=i+1}^{n-1}(n^2 - 3n - 2nj + j^2 + 3j + 2)$.

We find that they are equal respectively to

$$s_1' = \frac{1}{3}(-3n^2i - 6n^2 + 3i^2n + 12ni + 11n + n^3 - i^3 - 6i^2 - 11i - 6),$$

$$s_2' = \frac{1}{6}(n^3 - 6n^2 + 11n - 3n^2i + 12ni - 11i - 6 + 3i^2n - 6i^2 - i^3).$$

Now, adding those expressions to rule out the sum $s_1 + s_2 = \sum_{i=0}^{n-4}(s_1' + s_2')$ leads easily to $\#(Q_n') = \frac{1}{8}n(n^3 - 6n^2 + 11n + 2)$.

The adaptation of these calculations to compute $\sharp(Q_n'')$ leads to intractable computations depending on the parity of n and we choose instead a distinct approach. To do this, we observe that the positive integers $q_n' - q_n'' = \#(Q_n') - \#(Q_n'')$ enumerate admissible quadruplets (i, j, k, ℓ) in (Q_n') for which the straight-lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are parallel.

Let us consider the clique $\mathcal{K}_n$ associated to $\mathcal{C}_n$ with vertices set $E(\mathcal{K}_n) = \{z_m\}_{0 \le m \le n-1}$. This is a geometric graph that we may convert in a digraph, i.e. a directed graph, in $2^{\frac{1}{2}n(n-1)}$ ways. Let us choose some orientation among them, giving rise to a transitive tournament. Let $\phi: E(\mathcal{K}_n) \to [0, n[$ be the 1-to-1 mapping sending each vertex z_m to its rank $\phi(m)$ in a complete tournament on the vertices, so that an edge (z_i, z_j) appears if and only if $\phi(z_i) < \phi(z_j)$.

We consider a regular N-gon with $N = \frac{1}{2}n(n-1)$ vertices, being pairs $\{z_i, z_j\}$, with indices $0 \le i < j \le n - 1$.

Let $E\left(\mathcal{K}_n^{(2)}\right)$ be this set of pairs $\{z_i, z_j\}$. A choice of the mapping ϕ corresponds to an unambiguously defined mapping $\psi: E\left(\mathcal{K}_n^{(2)}\right) \to [0, N[$ given by $\psi(z_i, z_j) \to \phi(z_i) + \phi(z_j)$. In order to define a tournament on the set $E\left(\mathcal{K}_n^{(2)}\right)$, we must exclude inconsistent quadruplets $\{\{z_i, z_j\}, \{z_k, z_\ell\}\}$, for which $\psi(z_i, z_j) = \psi(z_k, z_\ell)$. So, inconsistent triplets, for which $\#\{z_i, z_j, z_k, z_\ell\} = 3$, must be removed. This being, an ordering may be chosen on the vertices set, by setting $\{z_i, z_j\} \preccurlyeq \{z_k, z_\ell\}$ when $\psi(z_i, z_j) < \psi(z_k, z_\ell)$. This sequence corresponds to the number of binary strings of length $n + 1$ with exactly one pair of adjacent 0's and exactly two pairs of adjacent 1's, and constitutes the entry A080838 in Sloane (2023). It results therefore that the number of inconsistent quadruplets is equal to $\frac{1}{16}n(2n^2 - 8n + 7 + (-1)^n)$. With this preparation, one has $\#(Q_n'') = \frac{1}{16}n(2n^3 - 14n^2 - (-1)^n + 30n - 3)$, the proof being nothing but substracting $\#(Q_n') - \#(Q_n'')$ from $\#(Q_n')$ computed earlier.

It remains to prove that $N_n \leq n + \sharp(Q''_n)$. We recast Lemma 6 as follows: Let (i,j,k,ℓ) so that $0 \leq i < j < n$, $i < k < \ell < n$, $k + \ell - i - j \notin \{-n, 0, n\}$. Then

$\sharp\{i,j,k,\ell\} = 3 \Leftrightarrow |z_{i,j,k,\ell}| = 1 \Leftrightarrow z_{i,j,k,\ell} \in \mathcal{C}_n \Leftrightarrow (i,j,k,\ell) \notin Q'_n$.

Indeed, under the assumptions of Lemma 6, the intersection point $z_{i,j,k,\ell}$ exists. The equivalences between the three first statements are a consequence of Lemma 6 together with its proof while the equivalence between the fourth and the first statements is easy. Now, $\sharp(Q''_n)$ is an upper bound of $N_{e,n} + N_{i,n}$ in view of the contraposition. Adding the n points located on $\mathcal{C}_n$ we obtain the upper bound mentioned in the proposition. This ends the proof of the Proposition. □

The sequence $q'_n = n + \#(Q'_n)$ begins as follows

Table 6. *Upper-bound for Number of Intersection Points*

n	3	4	5	6	7	8	9	10	11	12	13	14	...
q'_n	3	7	20	51	112	218	387	640	1001	1497	2158	3017	…

The numbers of lost points of intersection regarding to parallel straight lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ begins as

Table 7. *Upper-bound for Number of Lost Points due to Parallelism*

n	3	4	5	6	7	8	9	10	11	12	13	14	...
$q'_n - q''_n$	0	2	5	12	21	36	54	80	110	150	195	252	…

These tables allow to recover the tables given in the introduction. When we divide the numbers $q'_n - q''_n$ (except the very first one) by the number $n \geq 1$ of points in $\mathcal{C}_n$, one obtains a new interesting sequence of integers, namely

0, 0, 1, 2, 3, 4, 6, 8, 10, 12, 15, 18, 21, 24, 28,…

This last sequence is made of successive blocks of four increasing integers in arithmetic progression $(a, a + b, a + 2b, a + 3b)$, with a ratio b belonging itself to the arithmetic progression made by the integers ℕ while the starting point a of each quadruplet belongs to a sequence of shifted triangular numbers.

Let us discuss now the multiplicities and the directions of parallelism of the clique arrangement. We refer to the works of Wetzel (1978) and Ryckelynck and Smoch (2023) for the background and definitions.

Proposition 4.

1. The multiplicity of the origin is 0 if n is odd and $\frac{n}{2}$ if n is even.
2. The number of quadruplets $(i,j,k,\ell) \in Q_n$ such that $z_{i,j,k,\ell} = 0$ is 0 if n is odd and $\binom{\frac{n}{2}-1}{2}$ if n is even.
3. The multiplicity of $z_m \in \mathcal{C}_n$ is $n - 1$.
4. The number of quadruplets $(i,j,k,\ell) \in Q_n$ such that $z_{i,j,k,\ell} = z_m$ is $\binom{n-1}{2}$.

5. The number of directions for which more than one parallel occurs is equal to n.
6. The number of quadruplets $(i, j, k, \ell) \in \mathcal{Q}_n$ such that the lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are parallel is equal to $\binom{n}{2}$.

Proof.

1. The multiplicity of the origin is the number of couples of straight lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ with $\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell} = 0$. But when n is odd, if $z_m \in \mathcal{C}_n$ for some index $m \in \{0, \dots, n-1\}$ then $-z_m \notin \mathcal{C}_n$ since $(-z_m)^n = (-1)^n = -1$. When n is even, all the $\frac{n}{2}$ straight lines $\mathcal{D}_{i,\mathrm{i}+\frac{n}{2}}$ are concurrent through the origin.
2. The number of quadruplets (i, j, k, ℓ) such that $z_{i,j,k,\ell} = 0$, i.e. such that z_i and z_j (respectively z_k and z_ℓ) are symmetric with respect to the origin, is the number of quadruplets of the shape $\left(i, i + \frac{n}{2}, k, k + \frac{n}{2}\right)$, with $0 \leq i < k < \frac{n}{2}$ whence the formula.
3. For all $m \in \{0, \dots, n-1\}$, the $n-1$ straight-lines $\mathcal{D}_{i,m}$ are concurrent through the point z_m.
4. The quadruplets we are looking for are congruent modulo $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ to quadruplets of the shape (m, i, m, j) where $0 \leq i < j < n$ and $i \neq m, j \neq m$ whence the result.
5. Let us fix an integer i. All quadruplets of the shape $(i, i+1, i-k, i+1+k)$ are such that the straight lines $\mathcal{D}_{i,i+1}$ and $\mathcal{D}_{i-k,i+1+k}$. Let us consider the two mid-points $\frac{1}{2}(z_i + z_{i+1})$ and $\frac{1}{2}(z_{i-k} + z_{i+1+k})$. The line passing through these mid-points is the bisection of the segments $[z_i, z_{i+1}]$ and $[z_{i-k}, z_{i+1+k}]$ and thus is perpendicular to these segments. So the lines supporting them are parallel.
6. Since we have $i \in \{0, \dots, n-1\}$ and $k \in \{0, \dots, (n-1)/2\}$, the result holds. □

The Galoisian Properties of the Values of Squared Moduli $J_n(i, j, k, \ell)$

The cyclotomic field $\mathbb{Q}(\zeta_n)$ is a galoisian extension of the rationals $\mathbb{Q}$ of degree $\varphi(n)$ with ring of integers $\mathbb{Z}[\zeta_n]$ and with Galois's group $(\mathbb{Z}/n\mathbb{Z})^*$ of which the operation on $\mathbb{Q}(\zeta_n)$ is done through exponentiation $\zeta_n \to \zeta_n^g$. We denote as usual by $\Phi_n(z) \in \mathbb{Z}[z]$ the cyclotomic polynomial of index n and degree $\varphi(n)$, which is the minimal polynomial of ζ_n and whose roots are all primitive roots of unity.

In this section we give formulas for $J_n(i,j,k,\ell)$ and $J'_n(i,j,k,\ell)$ in the field $\mathbb{Q}(\zeta_{2n})$. In other words we concentrate on formulas such as those given in the following result:

Proposition 5. For all integers $n \geq 1$ there exists a smaller integer b_n such that for all $(i,j,k,\ell) \in Q''_n$, one may find a vector of integers (λ_m) such that one has the formula $J_n(i,j,k,\ell) = \frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1} \lambda_m \zeta_n^m$.

Proof. Let us fix n and put $\zeta = \zeta_{2n}$. For all k, we have $\cos\left(k\frac{\pi}{n}\right) = \cos\left(k\frac{2\pi}{2n}\right) = \frac{1}{2}(\zeta^k + \zeta^{-k}) = \frac{1}{2}(\zeta^k + \zeta^{2n-k})$. So, $\cos^2\left(k\frac{\pi}{n}\right) = \frac{1}{4}(\zeta^{2k} + \zeta^{2n-2k} + 2)$.

Hence if $(i,j,k,\ell) \in Q''_n$, and if we set $p = j - i, q = \ell - k, r = k + \ell - i - j$, (with $p + q + r$ an even positive integer) we get

$$J_n(i,j,k,\ell) = \frac{\cos^2\left(p\frac{2\pi}{2n}\right)+\cos^2\left(q\frac{2\pi}{2n}\right)-2\cos\left(p\frac{2\pi}{2n}\right)\cos\left(q\frac{2\pi}{2n}\right)\cos\left(r\frac{2\pi}{2n}\right)}{1-\cos^2\left(r\frac{2\pi}{2n}\right)}$$

and so $J_n(i,j,k,\ell)$ is equal to

$$\frac{4 + \zeta^{2p} + \zeta^{2n-2p} + \zeta^{2q} + \zeta^{2n-2q} - (\zeta^p + \zeta^{2n-p})(\zeta^q + \zeta^{2n-q})(\zeta^r + \zeta^{2n-r})}{2 - \zeta^{2r} - \zeta^{2n-2r}}.$$

This is the evaluation at $z = \zeta_{2n}$ of the rational function N/D where we have set

$N = 4 + z^{2p} + z^{-2p} + z^{2q} + z^{-2q} - z^{r+p+q} - z^{r+p-q}$
$-z^{r-p+q} - z^{r-p-q} - z^{-r+p+q} - z^{-r+p-q} - z^{-r-p+q} - z^{-r-p-q}$, and
$D = 2 - z^{2r} - z^{2n-2r}$.

Let us fix $(i,j,k,\ell) \in Q''_n$ and let us denote p, q, r the positive integers
$p = j - i\ ,\ q = \ell - k\ ,\ r = k + \ell - i - j.$

We noting that $p + q + r$ is even. We introduce integers $(\lambda_m)_{0\leq m<\varphi(2n)}$ and a common denominator b_n (all these integers being at the time indetermined) and we define the polynomial with rational coefficients

$$P(z) = \frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1} \lambda_m z^m.$$

With p, q, r, b_n being fixed, let us show how to characterize in a unique way the sequence (λ_m) so that the two polynomials

$b_n(4 + z^{2p} + z^{-2p} + z^{2q} + z^{-2q} - z^{r+p+q} - z^{r+p-q} - z^{r-p+q} - z^{r-p-q} - z^{-r+p+q} - z^{-r+p-q} - z^{-r-p+q} - z^{-r-p-q})$
,
and $(2 - z^{2r} - z^{2n-2r})\sum_{m=0}^{\varphi(2n)-1} \lambda_m z^m$, are congruent modulo the ideal generated in the Laurent algebra of rational functions $\mathbb{Q}[z, z^{-1}]$ by the cyclotomic polynomial $\Phi_{2n}(z)$.

To obtain a matricial system for the vector $\frac{1}{b_n}(\lambda_m)$, we compute the remainder of the euclidean division of the numerator of $N(z)/D(z) - P(z)$ by the cyclotomic polynomial $\Phi_{2n}(z) \in \mathbb{Z}[z]$, and require vanishing of all its coefficients. This determines uniquely the coefficients $\frac{1}{b_n}(\lambda_m)$. Now, the computation of b_n is nothing but searching the g.c.d. of all denominators of all characteristic vectors. □

The following corollary gives an application to Chebyshev polynomials of the first kind.

Proposition 6. For any integer $n \geq 3$ and (p, q, r) with $p + q + r$ even, if (λ_m) is defined as in Proposition 5, the polynomial

$$b_n\left(T_p(x) + T_q(x) - 2T_p(x)T_q(x)T_r(x)\right) - (1 - T_r(x)^2)\sum_{m=0}^{\varphi(2n)-1} \lambda_m\, x^m$$

is a multiple of the cyclotomic polynomial $\Phi_n(x)$.

Proof. Let us choose any quadruplet (i, j, k, ℓ) such that Proposition 5 holds. This is possible since $p + q + r$ is even. Since we have $T_k(\cos\theta) = \cos(k\theta)$ for any integer k and any real , we get

$$J_n(i, j, k, \ell) = \frac{T_p(\alpha) + T_q(\alpha) - 2T_p(\alpha)T_q(\alpha)T_r(\alpha)}{1 - T_r(\alpha)^2}$$

where $\alpha = \cos\left(\frac{\pi}{n}\right)$.

The rational fraction $\left(T_p(x) + T_q(x) - 2T_p(x)T_q(x)T_r(x)\right)/\left(1 - T_r^2(x)\right)$ and the polynomial $P(x) = \frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1} \lambda_m\, x^m$ agree at the point $x = \alpha$ in view of the previous proposition. So the polynomial in the claim is a multiple of the minimal polynomial of α. □

Remark 1. To compute the lattice vectors λ, one may devise the following algorithm. First, obtain g.c.d. between the numerator and the denominator of the fraction occuring in Proposition 6, next write the long division of this fraction, and at last, write a Bezout identity involving Chebyshev and cyclotomic polynomials. In this respect, we may refer to the work of Rayes et al. (2005) for properties like Bezout identities, resultants, g.c.d., for the family of polynomials (T_k). Nevertheless, the extension of those results to the required family of polynomials used in the previous procedure is very complicated to handle.

We continue this section with the study of the action of the Galois's group of $\mathbb{Q}(\zeta_{2n})$ on the sets Q_n' and Q_n''. If σ is an automorphism of $\mathbb{Q}(\zeta_{2n})$ then there exists an integer g prime to $2n$ such that one has $\sigma(\zeta_{2n}) = \zeta_{2n}^g$ and conversely. Thus we have an isomorphism of groups $\sigma: g \to \sigma_g$ and we get $\sigma_g\left(\zeta_{2n}^h\right) = \zeta_{2n}^{g^h}$

for all integer $h \geq 0$. The following proposition is intended to transfer the action of the Galois group on the field to an action on the sets of admissible quadruplets.

Proposition 7. The Galois group $(\mathbb{Z}/2n\mathbb{Z})^*$ of $\mathbb{Q}(\zeta_{2n})$ acts faithfully on Q_n as follows. Let $g \in (\mathbb{Z}/2n\mathbb{Z})^*$ and $(i,j,k,\ell) \in Q_n$ then $\sigma_g(i,j,k,\ell) \equiv (i',j',k',\ell')$ modulo $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with
$i' \equiv i$, $j' - i' \equiv g(j-i)$, $\ell' - k' \equiv g(\ell - k)$,
$k' + \ell' - i' - j' \equiv g(k + \ell - i - j) \bmod n$.
Each transformation σ_g preserves Q_n' and Q_n''. The orbits under the action of the Galois group have cardinalities divisors of $\varphi(2n)$. The conjugates of values of J_n (resp. $\tilde{J}_n$) are themselves values of J_n (resp. $\tilde{J}_n$). The mappings J_n and $\tilde{J}_n$ are contravariant with respect to these actions, i.e., $\sigma \circ J_n = J_n \circ \sigma$ for all $\sigma \in (\mathbb{Z}/2n\mathbb{Z})^*$. The Galois group acts as permutation group on the set of lattice vectors of radii cyclotomic representations.

Proof. First, we prove that the equations given in Proposition 7 determines without any ambiguity the quadruplet (i,j,k,ℓ). If we replace congruence modulo n by stricto sensu equalities, and if we use the triplet (p,q,r), the transformation becomes $p' = gp$, $q' = gq$, $r' = gr$ with $p' + q' + r' = g(p+q+r)$ even. In this way, we obtain the solution as
$(i',j',k',\ell') \equiv (i, i+gp, i+g(k-i), i+g(k-i)+gq)$,
modulo $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.. Obviously, σ_g preserves by definition the set Q_n' and since when $r \not\equiv 0 \bmod n$, we have also $gr \not\equiv 0 \bmod n$, we see also that σ_g preserves also Q_n'' in view of Gauss lemma. The formulas of Proposition 7 give rise to an equivalence relation since reflexivity occurs by choosing $g = 1$, symmetry holds by using g' such that $gg' \equiv 1 \bmod 2n$, and transitivity holds with $g'' = gg'$ which is prime to $2n$. By Lagrange's theorem on finite groups, each equivalence class has cardinality being a divisor of the order of $(\mathbb{Z}/2n\mathbb{Z})^*$ which is $\varphi(2n)$. The previous formulas show that orbits have maximal cardinalities $\varphi(2n)$ when $j-i$, $k-i$ and $\ell - k$ are prime to n .

Let $g \in (\mathbb{Z}/2n\mathbb{Z})^*$ to which we associate an automorphism σ in the Galois group. For any quadruplet (i,j,k,ℓ) we may compute the conjugate $\sigma\big(J_n(i,j,k,\ell)\big)$ as follows. Let us introduce the three integers

$$p = j - i, q = \ell - k, r = k + \ell - i - j,$$

and the two polynomials:
$w' = \zeta^{2gp} + \zeta^{2gn-2gp} + \zeta^{2gq} + \zeta^{2gn-2gq}$,
$w'' = (\zeta^{gp} + \zeta^{2gn-gp})(\zeta^{gq} + \zeta^{2gn-gq})(\zeta^{gr} + \zeta^{2gn-gr})$.
Then one obtains

$$\sigma\big(J_n(i,j,k,\ell)\big) = \frac{4 + \mathrm{w}' - \mathrm{w}''}{2 - \zeta^{2gr} - \zeta^{2gn-2gr}}$$

Expanding numerator and denominator shows that

$\sigma\big(J_n(i,j,k,\ell)\big) = J_n(i',j',k',\ell')$,

where the quadruplet (i',j',k',ℓ') is given by equations in Proposition 7. Now we deal with lattice vectors of radii cyclotomic representations. If $J_n(i,j,k,\ell) = \frac{1}{b_n}\sum_{m=0}^{n-1}\lambda_m\,\zeta^m$, we obtain $\sigma\big(J_n(i,j,k,\ell)\big) = \frac{1}{b_n}\sum_{m=0}^{n-1}\lambda_m\,\zeta^{gm}$. This implies that the lattice vectors of J_n and $\sigma\circ J_n$ are permuted vectors in $\mathbb{Z}^{\varphi(2n)}$. The remainder of the proof is straightforward. □

Proposition 8. The number of orbits with rational value of $J_n(i,j,k,\ell)$ is at least equal to the remainder of euclidean division of N_n by $\varphi(n)$.

Proof. Let us recall that the field $\mathbb{Q}(\zeta_{2n})$ is a galoisian extension of $\mathbb{Q}$, and, for any $\sigma \in (\mathbb{Z}/2n\mathbb{Z})^*$, σ generates $(\mathbb{Z}/2n\mathbb{Z})^*$ so σ has no fixed point except rational numbers. Otherwise, the subfield of invariant of σ won't be equal to $\mathbb{Q}$, contrarily to Galois's theorem. So, for any $(i,j,k,\ell)\in Q_n''$, $J_n(i,j,k,\ell)$ is irrational if and only if the equivalence class of $J_n(i,j,k,\ell)$ modulo the Galois's group has cardinality $\varphi(2n)$, or, translated in the action of $(\mathbb{Z}/2n\mathbb{Z})^*$ over Q_n'', the quadruplet (i,j,k,ℓ) spans an orbit of cardinality $\varphi(2n)$. Similarly, rational values of $J_n(i,j,k,\ell)$ give rise to distinct equivalence class in Q_n'' modulo $(\mathbb{Z}/2n\mathbb{Z})^*$ with cardinality 1. So, if $M_{r,n}$ (respectively $M_{s,n}$) denotes the number of rational (respectively irrational, with surds) values of $J_n(i,j,k,\ell)$, one has $\sharp(Q_n'') = M_{s,n}\varphi(2n) + M_{r,n}$. This implies our statement. □

Now, we conclude with concrete situations giving rise to explicit cyclotomic representations of the square radii. For the sake of brevity, we give only three examples, for n equal to 7, 8, 9.

Example 1. Let us consider the case $n=7$ where $\varphi(14)=6$ and $\Phi_{14}(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$. Let $\zeta = \zeta_{14}$ be a primitive root of unity of order 14, satisfying $\Phi_{14}(\zeta) = 0$. Consider for instance the radius

$j = -2(\zeta^4 - 2\zeta^3 + 3\zeta^2 - 2\zeta + 1)\cdot\zeta^{-1}\cdot(\zeta^2 - 2\zeta + 1)^{-1} \simeq 6.493959$.

Then the equations for the sequence $\lambda = (\lambda_5, \dots, \lambda_0)$ are

$\lambda_0 + \lambda_3 - 2\lambda_4 = 4, \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 = 0, -\lambda_3 + \lambda_4 + \lambda_5 = -2,$

$\lambda_1 - 2\lambda_2 + 2\lambda_4 - \lambda_5 = -2, -2\lambda_0 + \lambda_1 - \lambda_3 + 2\lambda_4 - \lambda_5 = -6,$

$\lambda_0 - 2\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4 + \lambda_5 = 4.$

Solving we find what we call the characteristic vector $\lambda = (\lambda_6, \dots, \lambda_0) \in \mathbb{Z}^6$ of the radius j, and in this case it is precisely $(\lambda) = (-2,0,0,2,0,4)$. More generally, computing with Maple, we obtain $b_7 = 1$ and the following representations of the various squares of radii starting with $j_0 = 1$. Then we list them by increasing height, that is to say by increasing sum of absolute values of the coefficients of the representation. The height is a good and simple test to recognize that algebraic values are conjugates since in that case we deal with two vectors (λ_m) cyclically permuted and thus of same height.

For the following pairs of square of radii, we see that the components share the same integer vector (λ_m) and so are congruent by the automorphism $\sigma: \zeta_{14} \to \zeta_{14}^5$, with $g = 5$, $j_6 = 2\zeta^4 - 2\zeta^3 + 4$, $j_7 = -2\zeta^5 + 2\zeta^2 + 4$, with height 8. Of height 9 is $j_1 = 2\zeta^5 + \zeta^4 - \zeta^3 - 2\zeta^2 + 3$. Of height 10 are $j_6 = 2\zeta^5 - 2\zeta^4 + 2\zeta^3 - 2\zeta^2 + 2$ and $j_2 = \zeta^5 - 3\zeta^4 + 3\zeta^3 - \zeta^2 + 2$, but they are not conjugates for the vectors (λ_m) are not cyclically permuted. Of height 15 is $j_4 = -3\zeta^5 + 2\zeta^4 - 2\zeta^3 + 3\zeta^2 + 5$. Of height 20 now is $j_7 = 3\zeta^5 + 4\zeta^4 - 4\zeta^3 - 3\zeta^2 + 6$. The sequence ends with two squares of raddi with height 24, $j_8 = 4\zeta^5 - 7\zeta^4 + 7\zeta^3 - 4\zeta^2 + 2$, and with height 29, $j_9 = -7\zeta^5 + 3\zeta^4 - 3\zeta^3 + 7\zeta^2 + 9$.

Example 2. Now we deal with the case $n = 9$ where $\varphi(9) = 6$ and $\Phi_{18}(z) = z^6 - z^3 + 1$. Let $\zeta = \zeta_{18}$ be a primitive root of unity of order 18, satisfying $\Phi_{18}(\zeta) = 0$. Computing with Maple, we obtain $b_9 = 12$ and we may obtain the cyclotomic representations of the all squares of radii each multiplied by 12. Then we see that the distribution of orbits is as follows. First we get seven isolated such circles, with rational values of square of radii and of increasing heights from 12 to 54 with orbits having each 9 vertices. Six of these orbits are inside the regular nonagon C_9. For heights greater than 12, we obtain six further orbits. First of them, is of height 20 the radius $-4\zeta^5 + 4\zeta^4 + 12$, is of height 24 the radius $-4\zeta^5 + 4\zeta^2 + 4\zeta + 12$, is of height 32 the radius of an internal orbit given by $8\zeta^5 - 4\zeta^4 - 4\zeta^2 - 4\zeta + 12$, is of height 36 the radius $-4\zeta^5 - 4\zeta^4 + 8\zeta^2 + 8\zeta + 12$ is of height 48 the radius $-12\zeta^5 + 12\zeta^4 + 24$, and lastly, is of height 54 the radius $-12\zeta^5 + 3\zeta^4 + 9\zeta^2 + 9\zeta + 21$. It is quite interesting to note that the radius of the six external orbits are increasing w.r.t. ordinary ordering. Next come two internal orbits with conjugate and distinct radii $12\zeta^4 - 12\zeta^2 - 12\zeta + 24$, $12\zeta^5 - 12\zeta^2 - 12\zeta + 24$ of height 60, and then three other such conjugate external orbits with squared radii having lattice vectors defined by

$$(-12,24,-12,-12,48), (24,-12,-12,-12,48), (-24,12,12,12,48),$$

all having height 108. At last, we find thirteen other values, that we may list by increasing value of the height 120, 168, 192, 264, 264, 300, 303, 366, 408, 768, 852, 1020, 1164 of the square radii. Since heights are distinct, these 13 orbits are non conjugate, and thus all these quantitites are rational numbers. Among the orbits, six are internal, seven are external, and the ordering with numerical order is not respected.

Example 3. The last example is given by $n = 8$ where $\varphi(16) = 8$ and $\Phi_{16}(z) = z^8 + 1$. This example is far more complicated. Let $\zeta = \zeta_{16}$ be a primitive root of unity of order 16, satisfying $\Phi_{16}(\zeta) = 0$. We have $b_8 = 226$. We do not here provide the whole list of the sixteen squares of radii, but instead a few comments. If these squared radii are each multiplied by 226, the height are increasing and form the sequence:

0, 113, 226, 452, 452, 678, 886, 886, 886, 904, 904, 1338, 1338, 1808, 1808, 2658.

In this way we obtain three sets of conjugate circles, one of cardinal 3 and two others of cardinal 2. When two vectors $(\lambda_7, \dots, \lambda_0) \in \mathbb{Z}^8$ appear and are obviously permutations one of the other, then these lattice vectors are conjugate through a galoisian morphism. This situation holds on six orbits forming three group of two orbits, defined help to the characteristic vectors $(\lambda_7, \dots, \lambda_0) \in \mathbb{Z}^8$:

$$(0,0,0,0,113,0,-113,226), (0,0,0,0,-113,0,113,226),$$
$$(0,0,0,0,-226,0,226,452), (0,0,0,0,226,0,-226,452),$$
$$(0,113,0,-226,-452,113,452,452), (0,113,0,-226,452,113,-452,452).$$

Orbits with cyclotomic vectors $(\lambda_7, \dots, \lambda_0) \in \mathbb{Z}^8$ where small changes of signs are done appear also by pairs and are not conjugate. This is the case of the three pairs

$$(-68,-8,132,-144,116,120,-172,578),$$
$$(68,-8,-132,-144,-116,120,172,578),$$
$$(204,-24,-396,-432,-348,360,516,378),$$
$$(-204,-24,396,-432,348,360,-516,378),$$
$$(68,-8,-132,-144,-116,120,172,126),$$
$$(-68,-8,132,-144,116,120,-172,126),$$

and they constitute strange gifts of the computation. We note also that the height takes three times the value 886 without giving rise to three conjugate orbits, only two of them being conjugates. This being so, we may analyze in the case $n = 8$ the geometry of this clique arrangement help to the explicit values:

$$0,1,4 \pm \sqrt{5}, 5 \pm 2\sqrt{5}, \frac{1}{2} \pm \frac{1}{10}\sqrt{5}, \frac{3}{2} \pm \frac{1}{2}\sqrt{5},$$
$$\frac{7}{2} \pm \frac{3}{2}\sqrt{5}, \frac{7}{2} + \frac{1}{2}\sqrt{5}, 6 + \sqrt{5}, 11 + 4\sqrt{5}, \frac{3}{8} - \frac{1}{8}\sqrt{5}.$$

We observe in this case the appearance of radii of which some conjugates are removed. This is easily understood and arise because some orbits of Q'_8 have non void intersections together with Q''_8 and $Q'_8 - Q''_8$.

Remark 2. Let $\xi_n = 2\cos\left(\frac{2\pi}{n}\right) = \zeta_n + \zeta_n^{-1}$ and let $\mathbb{Q}(\xi_n) = \mathbb{Q}(\zeta_n) \cap \mathbb{R}$ be the maximal real subfield contained in the field $\mathbb{Q}(\zeta_n)$ and generated by ζ_n. The minimal polynomial $\Psi_n(z) \in \mathbb{Z}[z]$ of ξ_n over the rationals has been explicitly computed by Lehmer (1933). For $n \geq 3$, it is a non-zero monic polynomial of degree $(\varphi(2n) - 1)/2$ with integer coefficients. The preceeding formulas for $x_{i,j,k,\ell}, y_{i,j,k,\ell}, J_n(i,j,k,\ell), J'_n(i,j,k,\ell)$, in both cases of Lemma 1 and Lemma 2, show that all these quantities are algebraic numbers belonging to $\mathbb{Q}(\xi_{2n})$. As a consequence, they may be written under the form $\frac{1}{b_n}\sum_{m=0}^{(\varphi(2n)-1)/2} a_m \, {\xi_n}^m$, for convenient and uniquely defined integers a_m, depending on i,j,k,ℓ and b_n depending only on n.

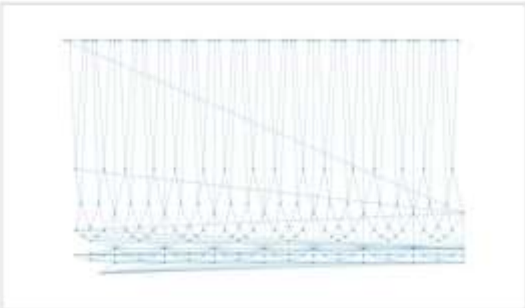
Since dealing with the non-monic polynomial $2^{\frac{1}{2}\varphi(2n)}\Psi_{2n}(z/2) \in \mathbb{Z}[z]$ closely related to Lehmer's polynomial is far from being easy, we do not take in account the reality of algebraic numbers.

Open Problems

In this section we conclude this work with some perspectives.

We may compare the clique arrangement $\mathcal{K}_n$, to which this paper is devoted, and the cyclotomic arrangement $\mathcal{R}_n$, see Ryckelynck and Smoch (2003), since their construction relies on $\mathcal{C}_n$. But the analogy ends there since geometrical, topological and combinatorial properties of $\mathcal{K}_n$ are far more complicated. This being so, we suggest as a first open problem the enumeration of the polygonal compact or non-compact chambers of $\mathbb{R}^2 - \mathcal{K}_n$ together with the description as it has been done for $\mathcal{R}_n$. In our opinion, this problem is the main one. Even if Poonen and Rubinstein (1998) have provided formula for the number of regions included in the unit disk $|z| \leq 1$, they did not characterize their geometry nor than the number of regions outside the disk.

Figure 2. *Polar Representation of the Geometric Graph of the Clique for* $n = 8$



It is quite surprising that the chambers of the cyclotomic arrangement $\mathcal{R}_n$ are of simple shapes, triangles or quadrilaterals or pentagons, compact or non compact. Experimentation with matlab shows that polygonal compact chambers of $\mathcal{K}_n$ with 6, 7 or 8 sides appear for examples with n smaller than 13. We propose as a difficult problem to prove that any number k of sides is admissible, i.e., for each integer k, there exists an integer n such that $\mathcal{K}_n$ admits at least one polygonal chamber with k sides.

Let us discuss one important step in the preceding program linked to the connectivity of the geometric graph $\mathcal{K}_n$. We must deal at first with the adjacency relation in the set of vertices in this geometric graph. If $z_{i,j,k,\ell}$ and $z_{i',j',k',\ell'}$ denote two intersection points, let us consider their connectivity $z_{i,j,k,\ell} \sim z_{i',j',k',\ell'}$. The straight line joining these two intersection points must coincide necessarily with two of the four straight lines $\mathcal{D}_{i,j}$, $\mathcal{D}_{i',j'}$, $\mathcal{D}_{k,\ell}$ or $\mathcal{D}_{k',\ell'}$ so we have for instance

$(i,j) = (i',j')$. Next, the abscissas of $z_{i,j,k,\ell}$ and $z_{i',j',k',\ell'}$ along to $\mathcal{D}_{i,j}$ must be close one to the other in the sense that no other abscissa of $z_{i,j,k',\ell'}$ lies between the two previous ones.

Experimentations with $\mathcal{K}_n$ with $n = 3, \ldots, 12$, have led us to find explicit boolean adjacency matrices $\mu_n \in \mathcal{M}_{N_n}$ of a shape, in a way, similar to the adjacency matrix that we presented in the case of $\mathcal{R}_n$. We propose as an open problem to find and prove such adjacency matrices. We suggest that the generation of μ_n relies on a lexicographic ordering of the vertices using first increasing values of the radii and next, increasing values of the polar angle along the orbits. To obtain this lexicographic order, one must convert $\mathcal{K}_n$ in a lattice in the euclidean plane by using polar coordinates as depicted in the figure above.

A third interesting problem concerns the computation of the multiplicities of the intersection points. To do that, one must solve pair of trigonometric equations $x_{i,j,k,\ell} = x_0$ and $y_{i,j,k,\ell} = y_0$ and convert them into diophantine ones. This seems a very difficult problem.

References

Bol G (1936) Beantwoording van prijsvraag no. 17. (Answer to competition no. 17.) *Nieuw Archief voor Wiskunde* (18): 14–66.

Harborth H (1969a) Diagonalen im regularen n-Eck. (Diagonals in regular n-disk.) *Elem. Math*. (24): 104–109.

Harborth H (1969b) Number of intersections of diagonals in regular n -gons, Combinatorial Structures and their Applications. In *Proceedings of Calgary International Conference,* 151–153. Calgary, Alta.

Lehmer DH (1933) A note on trigonometric algebraic numbers. *American Mathematical Monthly* 40(3): 165–166.

Poonen B, Rubinstein M (1998) The number of intersection points made by the diagonals of a regular polygon. *SIAM Journal on Discrete Mathematics* 11(1): 135–156.

Rayes MO, Trevisan V, Wang PS (2005) Factorization properties of Chebyshev polynomials. *Computers and Mathematics with Applications* 50(8): 1231–1240.

Ryckelynck P, Smoch L (2023) On cyclotomic arrangements of lines in the plane. *International Journal of Geometry* 12(2): 70–88.

Rigby JF (1980) Multiple intersections of diagonals of regular polygons, and related topics. *Geometriae Dedicata* 9(Jun): 207–238.

Sloane N (2023) *The on-line encyclopedia of integer sequences*. The OEIS Foundation Inc.

Steinhaus H (1958) Problem 225. *Colloq. Math*. 5.

Steinhaus H (1983) *Mathematical snapshots*. Oxford University Press.

Tripp CE (1975) Adventitious angles. *Math. Gazette* (59): 98–106.

Washington LC (1997) *Introduction to cyclotomic fields*. Springer-Verlag, G.T.M. 83.

Wetzel JE (1978) On the division of the plane by lines. *The American Mathematical Monthly* 85(8): 647–656.

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Teaching Mathematics with Visuals

*By Gohar Marikyan**

This article is about research done on the ways of using visuals in teaching introductory mathematics. Research showed that the use of visuals improves the learning of mathematics. However, more research is needed to explore which math concepts need to be addressed, how to choose visuals for that purpose, and how to incorporate them in teaching mathematics. The analysis shows that the effective use of correctly chosen visuals in teaching introductory mathematics positively impacts students' attitudes toward mathematics, and enhances their learning, engagement, and motivation.

Keywords: *analytical thinking skills, instructional strategies with visuals, introductory mathematics, student engagement, and motivation*

Introduction

The acquisition of analytical thinking skills is an essential component of learning introductory mathematics. However, many students struggle with this aspect of mathematics education, which can lead to negative attitudes toward the subject. One potential solution is the use of visuals to support the development of analytical thinking skills. Visuals have been found to be effective in promoting mathematical thinking, problem-solving, and reasoning skills.

This article explores the use of visuals in developing students' analytical thinking skills, by exploring how sticks and abacus can be effectively used in teaching arithmetic. The purpose of this study is to investigate the impact of using visuals in mathematics instruction on the development of analytical thinking skills. The article begins with a review of relevant literature, including theoretical frameworks and previous research on the use of visuals in education and their impact on analytical thinking skills development.

The article concludes with thoughts, recommendations, and discussions for using visuals to support the development of analytical thinking skills. The significance of this study lies in its potential to inform instructional practices that promote student success in mathematics, especially in the development of analytical thinking skills.

Previous Research on Visuals in Education

In recent years, there has been increasing interest in the use of visuals to support learning in various subject areas. This interest is driven by the recognition that visuals can enhance students' understanding, comprehension, and retention of

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concepts, as well as their problem-solving and reasoning skills.

Previous research has demonstrated the effectiveness of visuals in improving student learning outcomes in mathematics. For example, John Hattie's meta-analysis of educational research studies found that the use of visuals had a moderate to large effect size on student achievement across all academic subjects. Hattie’s study, which analyzed data from over 800 meta-analyses, concluded that the use of visual aids such as diagrams, graphs, charts, and pictures can help students better understand complex concepts and retain information more effectively. Additionally, the study found that teachers who incorporate visual aids into their lessons can help students develop stronger critical thinking skills and improve their problem-solving abilities (Hattie 2009).

Moyer-Packenham, Westenskow, and Jordan conducted a meta-analysis of the effects of visual representations on learning mathematics and found that visual representations had a positive effect on learning outcomes (Moyer-Packenham et al. 2016). Similarly, Suh, Moyer-Packenham, and Westenskow investigated the effects of visuals on problem-solving in elementary school mathematics and found that visuals improved students’ problem-solving skills (Suh et al. 2016).

Theoretical frameworks, such as cognitive load theory, also support the use of visuals in mathematics instruction. Cognitive Load Theory (CLT) suggests that instructional materials should be designed in a way that reduces the cognitive load on learners’ working memory. The use of visuals in mathematics instruction is one way to achieve this goal, as visuals can help learners better understand and process mathematical concepts (Kalyuga 2009). This can help students to process information more effectively and efficiently, leading to improved learning outcomes.

Other studies have investigated the impact of specific types of visuals on learning outcomes in mathematics education. For example, Tarmizi and Sweller (1988) found that the use of worked examples and diagrams improved learning outcomes in algebra.

Overall, these studies suggest that the use of visuals can be an effective strategy for improving learning outcomes in mathematics education. By providing students with visual representations of mathematical concepts and relationships, visuals can help students to make sense of complex information and develop the analytical thinking skills necessary for success in mathematics and other academic disciplines.

However, despite the potential benefits of using visuals in mathematics education, there is still a need for further research on the impact of visuals on analytical thinking development, especially in introductory mathematics students. This study seeks to address this gap in the literature by investigating the effectiveness of visuals in developing analytical thinking skills in mathematics education.

Purpose of the Study

The purpose of this study is to investigate the impact of using visuals in mathematics instruction on the development of analytical thinking skills in students. The study seeks to answer the following research question: What is the impact of using visuals in mathematics instruction on the development of analytical thinking skills in students?

The study aims to contribute to the understanding of how visuals can be used to support the development of analytical thinking skills in mathematics education. By investigating the impact of visuals on analytical thinking skills development, the study seeks to provide insights into instructional practices that promote student success in mathematics, especially in the development of analytical thinking skills.

Significance of the Study

The significance of this study lies in its potential to inform instructional practices that promote student success in mathematics, especially in the development of analytical thinking skills. Analytical thinking skills are crucial for success in mathematics and other academic disciplines, as well as for real-world problem-solving and decision-making. However, many students struggle with the development of analytical thinking skills, which can lead to negative attitudes toward mathematics and reduced academic performance.

The study's findings on the impact of visuals on analytical thinking skills development can provide valuable insights into how educators can support students' analytical thinking skills development. By identifying effective instructional strategies, such as the use of visuals, educators can enhance students' mathematical thinking and problem-solving abilities, leading to improved academic performance and increased confidence in mathematics.

Overall, the study's findings on the impact of visuals on analytical thinking skills development in mathematics education have the potential to make a significant contribution to improving students' academic outcomes and attitudes toward mathematics.

Theoretical Framework

This study is informed by several theoretical frameworks related to the use of visuals in mathematics education and the development of analytical thinking skills.

The first theoretical framework is the National Council of Teachers of Mathematics (NCTM) Principles and Standards for School Mathematics (2000), which emphasizes the importance of visualization in mathematics learning. The NCTM advocates for the use of visuals to help students make sense of mathematical concepts and relationships.

The second theoretical framework is cognitive load theory, which suggests that the use of visuals can help to reduce cognitive load.

The third theoretical framework is the conceptual change theory, which suggests that students' prior knowledge and beliefs can influence their learning of new concepts. The use of visuals can help students to develop new conceptual frameworks that are more accurate and better aligned with mathematical concepts and relationships.

The fourth theoretical framework is the problem-based learning theory, which emphasizes the importance of problem-solving and inquiry-based learning in mathematics education. The use of visuals can support problem-based learning by providing students with visual representations of problems and supporting their problem-solving skills.

Overall, these theoretical frameworks suggest that the use of visuals in mathematics education can support students' learning and development of analytical thinking skills. By reducing cognitive load, supporting conceptual change, and promoting problem-based learning, visuals can help students to make sense of mathematical concepts and relationships and develop the analytical thinking skills necessary for success in mathematics and other academic disciplines.

Use of Visuals in Developing Analytical Thinking Skills

The use of visuals can support the development of the analytical thinking skills of students in several ways.

First, visuals can support the development of spatial reasoning skills, which are essential for mathematical problem-solving. Visuals such as diagrams, graphs, and models can help students to visualize mathematical relationships and concepts, leading to improved spatial reasoning abilities (Clements and Sarama 2011).

Second, visuals can support the development of mathematical communication skills by providing a common visual language for students to communicate their mathematical thinking. This can help students to develop their ability to explain their thinking and ideas, leading to improved communication skills (Lannin and Barker 2015).

Third, visuals can support the development of metacognitive skills by encouraging students to reflect on their thinking and problem-solving strategies. Visuals can be used to prompt students to think about their thinking and reflect on their problem-solving processes, leading to improved metacognitive skills (Fyfe and McNeil 2018).

Overall, the use of visuals can support the development of analytical thinking skills in students by promoting spatial reasoning, critical thinking, mathematical communication, and metacognitive skills. By using visuals in mathematics education, teachers can help students to develop the analytical thinking skills necessary for success in mathematics and other academic disciplines.

The Impact of Visuals on Analytical Thinking Development

The methodology devised in Marikyan (2019) does not require students to spend hours on rote learning and memorization. The research showed that the only effective way of learning mathematics is through understanding. The net effect of learning mathematics through understanding is the development of students' analytical thinking skills. The research also showed students started being interested in mathematics, enjoyed learning it, and considerably progressed in learning mathematics (Marikyan 2019).

In mathematics, students are required to work with formulas and solve word problems. For both, and especially for solving word problems students need to use their analytical thinking skills. On the other hand, solving word problems develops analytical thinking skills. Therefore, teaching mathematics should nurture analytical thinking skills among students (Marikyan 2013).

However, unfortunately, there are required tests that all students have to pass, preferably with high scores. The aftermath is that the focus of learning mathematics becomes high test scores, and classrooms are being converted into "test preparation centers," then memorization becomes the fastest way of preparing students for the test. However, memorization only helps students to remember the topics to pass the test (Casbarro 2003). The net effect is that students are being checked and graded on memorization. Unfortunately, memorization may require a considerable amount of time for some students. Those students will be forced to memorize mathematics that does not make sense to them, since they do not understand it. Those students will make the same amount of effort to learn mathematics as to learn a poem by heart in a foreign language. Needless to say, this may create math anxiety among those students. Another group of students is those who are interested in understanding mathematics, but no one explains it to them because the teacher spends time mainly on preparing students for required tests with high grades. The frustration for these students is the lack of time and support, and they become discouraged from learning mathematics.

Another example is a child from a YouTube video who didn't want to go back to school because the teacher on Monday had said that $6 + 2 = 8$, on Tuesday she told that $4 + 4 = 8$, and on Wednesday she said that $5 + 3 = 8$. Then the child continues, saying that she'll wait until the "teacher makes up her mind." Sounds funny, but on a serious note, the teacher failed to explain why $6 + 2 = 8$, $4 + 4 = 8$, and $5 + 3 = 8$. This example shows that the child has analytical thinking skills which were not put to use in learning mathematics. She compared what the teacher taught without explaining and concluded that the teacher does not know what she was talking about. It is obvious that it could be very helpful to use visuals in explaining to the child why $6 + 2 = 8$, $4 + 4 = 8$, and $5 + 3 = 8$.

Below are two scenarios to compare. Both explain why $6 + 2 = 8$, $4 + 4 = 8$, and $5 + 3 = 8$.

Scenario 1. The teacher puts some sticks on the table and asks students to count 6 sticks and put them on a side. Then students count 2 sticks and add them to the other 6 sticks, and count all sticks again. Then using the same technique, the teacher teaches that $4 + 4 = 8$, and $5 + 3 = 8$. The result will be that the students memorize that $6 + 2 = 8$, $4 + 4 = 8$, and $5 + 3 = 8$. However, the students will not understand why they all are equal to 8, as the child in my above example.

Later on, the commutative property of addition will be introduced to students using the formula (1). The formula is not helpful because they had never used it before while adding numbers. Plus, students, having problems using variables will just ignore the formula, and the commutative property of addition will be never used in learning mathematics.

Scenario 2. The teacher puts 8 sticks on the table and asks students to count them. Now the students know that they have 8 sticks. Next, students are asked to take 6 sticks and put them on a side. Then students count the remaining 2 sticks. Students learn that taking away 6 sticks from 8 leaves with 2 sticks. Then the teacher asks them to add the 2 sticks to the 6 sticks, learning that 6 + 2 = 8. Next, the teacher asks students to separate 6 and 2 sticks and then add the 6 sticks to the 2 sticks, that is, 2 + 6 = 8. This teaches students that 6 + 2 = 8 and 2 + 6 = 8, teaching the concept of the commutative property of addition (1). The same activity can be repeated with 4 and 4, 5 and 3, and 1 and 7. Through this activity, students will learn the concepts of addition, subtraction, and the commutative property of addition (1).

$$a + b = b + a \qquad (1)$$

Explaining the commutative property of addition using sticks, at an early age, will prevent students from getting accustomed to performing addition through memorization.

Then using the same technique, the teacher teaches that 4+4=8, 5 + 3 = 8, and 7 + 1 = 8. The result will be that the students will learn that 6 + 2 = 4 + 4 = 5 + 3 = 7 + 1 = 8, and the commutative property of addition. The students will understand why they all are equal to 8.

The comparison of these two scenarios shows that although both scenarios used the same objects, the results were very different. This shows the importance of how visuals were used.

The associative property of addition (2) sometimes is more difficult to explain. Let us try to explain it using the same sticks.

$$(a + b) + c = a + (b + c) \qquad (2)$$

Scenario 3. The teacher puts the same 8 sticks on the table and asks students to divide them into three groups. Then students are asked to count the sticks in each group. Next, the teacher asks the students to combine the first and the second groups, then add the third group. The students know that the result is 8. Next, the teacher asks students to separate the same groups, combine the second and the third groups, and add them to the first group. From this activity, students will learn that combining three groups in any order will result in the same answer.

In the first grade in Armenia, children were required to have a box of 6-inch-long plastic sticks of the same color. They were used just for a few weeks to understand the concepts of addition and subtraction, and the commutative and associative properties of addition.

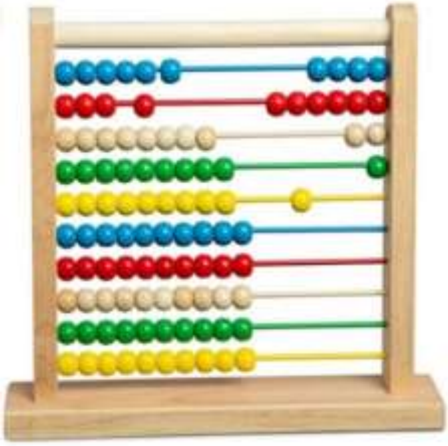
Using sticks, not other objects helped not to connect addition or subtraction with any particular objects. The sticks were of the same color which helped not to connect addition or subtraction with any particular color. Both are important for a smooth transition from concrete thinking to abstract thinking. Another net effect from using sticks of the same size and of the same color, later on, helps with the understanding of variables. This discussion is out of the scope of this article.

The first graders were also required to have an abacus. In fact, those two were the only visual aids used in teaching mathematics in Armenia.

The abacus in Figure 1 looks nice and may attract a child's attention.

However, this abacus can only be used to count from 1 to 10 and maybe can be used for adding small numbers, the sum of which is 10 or less.

Figure 1. *Colorful Abacus*



The abacus used in Armenia in the first grade was not as colorful as the one in Figure 1. The beads of that kind of abacus are of two colors. In each row, there are ten beads. The middle two beads of each row are of a contrasting color. In Figure 2 the middle two beads are black. The middle two beads are the fifth and sixth beads. That is, in each row (except the row with 4 beads) there are 4 white, 2 black, and another 4 white beads. This setup of beads helps students to minimize

Figure 2. *An Abacus Used by First Graders in Armenia*

Scenario 4. There is no need to count for moving 1 or 2 beads to the left. After moving 3 beads to the left, they see that out of 4 white beads only one is left, that is, 4 – 3 = 1. Once the students understand that 3 + 1 =4, they will move 3 beads to the left without counting 1, 2, 3, but will move the beads leaving 1 white bead, therefore, eliminating counting while moving 3 beads. To move 4 beads to the left

is not a brainer; those are the 4 white beads. To move 5 beads, students know that there are 4 white beads, therefore, they need to move one more bead, which is a black bead to get 5 beads. Then they see that half of the beads are now on the left side and the other half is on the right. Then students learn, that 5 + 5 =10, 5 × 2 = 10, and also, 10 ÷ 2 = 5. To move 6 beads to the left, they will move 5 beads and one more black bead to the left. The students learn that 5 + 1 = 6. Seeing 4 white and 2 black beads on the left, and 4 white beads on the right. Now they also learn that 4 + 2 = 6, 10 – 6 = 4, and 10 – 4 = 6. Again, no need to count the beads. While moving 7 beads, they will move the 4 white beads, the 2 black beads, and 1 more white bead. They learn that 6 + 1 = 7, 4 + 2 + 1 = 7. Seeing 3 white beads on the right side, they also learn that 10 – 7 = 3 and 10 – 3 = 7. This means, that for moving 7 beads to the left, they need to leave 3 white beads and move the rest to the left. Similarly, to move 8 or 9 beads to the left, they will move all but 2 or 1 bead, correspondingly.

Scenario 5. This abacus can also be used to teach place values. The third from the bottom is for ones, above which is the row of tens. Then comes the row of hundreds. The first bead of the next row is black, to help visually to easily find the row of thousands.

The row with 4 beads is a placeholder, similar to the decimal point. The lower two rows are for tenths and hundredths. For example, to show 0.75, 7 beads from the second from the bottom row and 5 beads from the last row will be moved to the left. (Tenths and hundredths were not covered in the first grade.)

The knowledge gained in Scenario 4, is applicable to all rows. That is, to calculate 5000 + 3000 students move 5 beads and 3 beads from the row of thousands to the left. Seeing 2 beads left on the right side they will write, 5000 + 3000 = 8000. If the students perform the same operation on the paper, they will write 0 + 0 = 0, 0 + 0 = 0, 0 + 0 = 0, 5 + 3 = 8 (See Figure 3).

The knowledge gained while using an abacus, students can use while performing calculations without an abacus.

Figure 3. *Step for Performing Addition on a Paper*

Step 1	Step 2	Step 3	Step 4
5000	5000	5000	5000
+	+	+	+
3000	3000	3000	3000
0	00	000	8000

The top row represents ten-thousands. How this abacus can be used in teaching mathematics?

Scenario 6. The teacher asks students to calculate 7 + 5 using the abacus. Students move 7 beads to the left. Then they move the remaining beads in the row, counting 1, 2, 3. Now all beads on the row of ones are on the left side, which means that 7 + 3 = 10. Students will move all 10 beads to the right and will move 1 bead from the row of tens to the left and will continue moving beads from the row of ones to the left, counting, 4, 5. Seeing 1 bead from the row of tens and 2 beads from the row of ones on the left, the student will know that 7 + 5 = 12. Besides learning that 7 + 5 = 12, the students learned that 5 = 3 + 2, more importantly, they learned that 12 is equal to one 10 and 2 ones.

Scenario 7. Similarly, to calculate 47 + 58, the students move 7 beads from the row of ones, and 4 beads from the row of tens to the left. Next, to add 58, the students move the remaining 3 beads from the row of ones, getting all ten beads on the left side. Moving the 10 beads to the right and moving 1 bead from the row of tens to the left, the students see that 47 + 3 = 50. That is 50 – 47 = 3. Then students move 5 beads from the row of ones to the left. They know that 47 + 8 = 55. The last step is to add 50, for which they move 5 beads from the row of tens to the left, getting all 10 on the right side. They move the 10 beads to the right and move 1 bead from the row of hundreds to the left. 47 + 58 = 105 because on the left there is 1 bead from the row of hundreds and 5 beads from the row of ones. While calculating 47 + 58, students also learned that 10 beads from the row of tens, make a hundred, that is, 10 × 10 = 100.

Scenarios 4 – 7 show that using this type of abacus in teaching mathematics helps students to learn some challenging concepts of mathematics. It also teaches how addition and subtraction are connected.

Discussion & Conclusion

The results of the study indicate that the use of visual aids, sticks and the abacus in teaching mathematics leads to a significant improvement in students' learning of mathematics, improving students' understanding and retention of learned mathematical concepts. The net effect of this improvement leads to the development of their analytical thinking skills.

Another result of this study is that the use of these visuals can help to make abstract or complex concepts more concrete and accessible to students and can also help to promote deeper learning and critical thinking skills.

Overall, these findings suggest that the use of these visuals can be an effective strategy for developing students' analytical thinking skills in mathematics. However, it is important to note that the specific types of visuals used, as well as the instructional strategies used to incorporate them in teaching, play an important role in their effectiveness.

This study did not explore how other visuals can be used in combination with other instructional strategies to further enhance students' analytical thinking skills in mathematics.

References

Casbarro J (2003) *Test anxiety & what you can do about it: a practical guide for teachers, parents, and kids*. Port Chester, NY: Dude Publishing, National Professional Resources.

Clements DH, Sarama J (2011) Early childhood mathematics intervention. *Science* 333(6045): 968–970.

Fyfe ER, McNeil NM (2018) Scaffolded feedback and metacognitive support promotes student self-regulation and advances mathematical problem solving. *Educational Psychology Review* 30(4): 1055–1073.

Hattie J (2009) *Visible learning: a synthesis of over 800 meta-analyses relating to achievement*. Routledge.

Kalyuga S (2009) Cognitive load theory and multimedia learning in mathematics. In MK Heid, GW Blume (eds.), *Research on Technology and the Teaching and Learning of Mathematics: Volume 2, Cases and Perspectives*, 61–76. Charlotte, NC: Information Age Publishing.

Lannin JK, Barker D (2015) Mathematics discourse in inclusive classrooms: a case study of one teacher's use of wait time and questioning strategies. *Journal of Education for Students Placed At Risk (JESPAR)* 20(1-2): 22–38.

Marikyan G (2013) *Development of logical think among students*. ATINER's Conference Paper Series, No. MAT2013-0474. Athens: ATINER.

Marikyan G (2019) Math anxiety, its development and elimination. In *International Conference on Social Science, Arts, Business and Education*. Venice, Italy, August 4-5.

Moyer-Packenham PS, Westenskow A, Jordan K (2016) A meta-analysis of the effects of visual representations on learning mathematics. *Educational Psychology Review* 28(4): 661–690.

Suh JM, Moyer-Packenham PS, Westenskow A (2016) The effects of visuals on elementary students' mathematics problem-solving. *Journal of Educational Research* 109(3): 271–280.

Tarmizi RA, Sweller J (1988) Guidance during mathematical problem solving. *Journal of Educational Psychology* 80(4): 424–436.

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The Availability Level of the Aesthetic Approach in Mathematics Textbooks for the Higher Grades at the Primary Stage in Saudi Arabia

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This study aims to establish the presence of aesthetic underpinnings in Saudi mathematics textbooks. For this, the researchers used descriptive-analytical methods. We looked at fourth, fifth, and sixth grade math textbooks. The arbitrated aesthetic method divides 27 signals into seven basic domains. The content availability of each main aesthetic approach area was found to be constant across all books in the elementary stage's upper grades. Aesthetic strategies were used in primary mathematics textbooks at a rate of 34%. The average percentages for the major domains were: 76.9% major themes, 57.7% mental level, 52.8% science framework, 35.4% mathematics and arts, 10.7% mathematics and emotional components, and 2.2 % athletic aesthetic criteria. The researchers suggested designing mathematics courses and texts artistically.

Keywords: *textbooks, mathematics, aesthetic approach, mathematical aesthetic, content analysis*

Introduction

Scholastic mathematics demonstrates mathematicians' accomplishments by studying significant mathematical concepts and operations, enables students to comprehend mathematics as a science of human intellectual accomplishments, and enables students to meet life requirements through observation of mathematics' application in the real world and enjoyment of its subjective and functional aesthetic.

The National Council of Mathematics Teachers in the United States of America's (NCTM) scholastic mathematics standards emphasized the necessity of assisting pupils in comprehending the essence of mathematics and appreciating its mathematical aesthetic (Asiri et al. 2013).

The aesthetic method is oriented to increase the enjoyment and joy associated with mathematics education. They identify aesthetics in knowledge, the means of accessing and verifying it, and the scientists' efforts to discover, interpret and manage scientific phenomena, and they believe that the aesthetic components elicit satisfaction and help students develop critical thinking abilities.

By incorporating mathematical aesthetic criteria into school mathematics curricula, the aesthetic in mathematics, with its patterns, numbers, shapes, geometric figures, algorithms, proofs, and explanations, becomes an introduction

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that presents mathematics as a vital subject exciting for research and curiosity and justifies and encourages its learning among students (Abdullah 2019).

Gadanidis et al. (2016) believe that aesthetic experiences in mathematics enable students to engage in research and investigation, unlock their imagination, and provide pleasure and variety to mathematics instruction.

Additionally, emphasizing the aesthetic components of mathematics instruction helps children acquire moral ideals. It promotes personality balance in students by connecting the beauty notion to the order, harmony, and consistency inherent in mathematics' beauty.

Sinclair (2009) asserted that an aesthetic approach to mathematics education impacts school mathematics education because it increases students' motivation, alleviates math anxiety, and promotes enjoyment of learning by exposing students to the nature of mathematics and its significance in their lives.

The aesthetic method elicits pleasure in pupils, improves their desire and curiosity about mathematics, and enables them to benefit from its tools. It improves their drive and curiosity for mathematics and benefits from the instruments utilized in daily life. Mathematics has an aesthetic component that should be zed in elementary school students' textbooks. It is founded on information and serves as a vehicle through which the curriculum designer communicates contemporary trends to pupils.

The Problem

The study problem arose from the aesthetic approach's role in achieving certain educational goals through student's practice of various thinking skills, research, and meditation within a framework of satisfying emotional aspects through aesthetic experiences, as confirmed by several studies: (Abdullah 2019, Saleh and Matalqa 2018, Mhenni 2018, Abu Zaid et al. 2017, Suleiman 2016, Minara 2017, Satyam 2016, Cellucci 2015, Sulaiman 2014, Abdul Hadi 2014).

Additionally to the recommendations made by the studies (Abdullah 2019, Ahmed 2018, Satyam 2016, Cellucci 2015, Abdul Hadi 2014), reconsidering the mathematics curriculum's content and placing emphasis on the aesthetic view of mathematics, as well as abandoning the traditional method, is necessary.

The Education and Training Evaluation Commission's national tests revealed students' weakness in mathematics in Saudi Arabian, with 41% of sixth-grade students failing to meet the minimum performance standards expected of them. Additionally, the International Study of World Trends in Academic Achievement in Mathematics and Science (TIMS 2019) revealed a decline in students' average performance in Saudi Arabia. These findings indicate that mathematics education has shortcomings that can be attributable to any educational process.

However, the Education Ministry recognized low-quality curricula as barriers to accomplishing the 2020 national transformation (Ministry of Education 2019). Because the majority of Arab education systems use textbooks as a curriculum (Al-Saadowi 2010), and because they serve as the official document that reflects the majority of curriculum components, there are observations about mathematics books revealed by the evaluation study of the mathematics and science project in Saudi Arabia prepared by the Center for Research Excellence in Science and Mathematics Development (2015). It advised aligning school mathematical standards with the nation's identity and culture and guaranteeing the best possible experiences during future growth processes.

As a result, the necessity for a review of school mathematics textbooks arises, and the study problem is defined as determining the level of aesthetic approach in mathematics textbooks for higher grades at the primary stage in Saudi Arabia.

Questions

The study questions are:

1. What aesthetic approach foundations must be available in mathematics books for the higher grades at the primary stage in Saudi Arabia?
2. What is the availability level of the necessary aesthetic approach foundations in mathematics books for the higher grades at the primary stage in Saudi Arabia?

Aims

The study aims to:

1. Building a list of the aesthetic approach foundations and indicators must be available in mathematics books for the higher grades at the primary stage in Saudi Arabia.
2. Identifying the availability level of the aesthetic approach foundations in mathematics books for the higher grades at the primary stage in Saudi Arabia.

Importance

The study importance appears in:

1. The current study lists the aesthetic approach foundations in mathematics, which may help designers develop mathematics curricula for the higher grades at the primary stage.
2. The study provides curriculum designers with tools and scientific criteria to evaluate mathematics books.
3. The study sheds light on the shortcomings in mathematics textbooks in Saudi Arabia, which may help curriculum developers to evaluate and develop them.

4. The study may help teachers take care of the aesthetic aspects of mathematics education.
5. The current study opens the way for researchers to conduct further research on the aesthetic approach in mathematics education.

The Limits

Objective limits: a list of the aesthetic approach foundations in the light of the content analysis of the mathematics books for the higher grades at the primary stage (fourth, fifth, and sixth) in Saudi Arabia in the academic year 1443 AH.
Temporal limits: Mathematics textbooks prescribed for primary school students, edition 1443 AH.
Spatial limits: Saudi Arabia.
Human limits: Students of the primary stage higher grades in Saudi Arabia.

Study Terminologies

The aesthetic approach: It is define as the building and implementing curricula to achieve educational goals and enjoy the technical and aesthetic aspects in science and emphasize the emotional and appreciation aspects that have often been neglected instead of its importance.

The researchers define it procedurally as the quantitative analysis of mathematics textbooks for the higher classes in light of the aesthetic approach foundations available in mathematics books.

Theoretical Framework and Previous Studies

This part deals with the specification of the aesthetic approach in mathematics in the light of its concept, its importance in mathematics, its foundations in mathematics textbooks, and the related previous studies.

The Aesthetic Approach in Mathematics

Aesthetic education aims to promote the human being since a focus on beauty necessitates developing and integrating students' personalities (Al-Sherbiny 2005). When students enjoy instruction and research due to their interactions with aesthetic experiences, they develop favorable attitudes toward science and learning and an appreciation for the role of scientists.

'Researchers' definitions of the aesthetic approach vary; it is defined as a proposal to develop and implement curricula to accomplish educational goals while also enjoying the technical and aesthetic aspects of science, as well as emphasizing the emotional and appreciation aspects of science that are frequently overlooked in favor of their importance.

Girod et al. (2010) define it as the experiences that individuals gain from their imagination seeking harmony in the strength and beauty of ideas and information that startle amaze the learner well as the splendor of these concepts and information.

Thus, the visual features of the entrance set it apart from other entries. The aesthetic method blends students' emotional, cognitive, and skill development, emphasizing emotional well-being and achieving pleasure in educational settings. The aesthetic method examines integrated phenomena and the whole awareness of the phenomenon to identify its beautiful characteristics.

The aesthetic approach to mathematics is consistent with mathematics' nature, as seen by the beauty of mathematical patterns and geometric shapes and their algorithms. The aesthetic approach to mathematics education connects mathematics to various disciplines and emphasizes its functional beauty in describing the situations, and issues students are exposed to.

The Importance of the Aesthetic Approach in Mathematics Education

It develops the integrated personalities of the students, refines their behavior, fosters their creative potential, provides emotional and spiritual fulfillment, and fosters favorable attitudes toward the environment and its beauty (Al-Sherbiny 2005).

Mathematics is a universal language. Because the world's symbols, numbers, and mathematical forms are nearly the same, the importance of mathematical beauty in creating universal moral qualities such as honesty, peace, respect for others' viewpoints, and social interaction is apparent (Minara 2017). The artistic perspective on classroom mathematics imbues it with meaning. It converts abstract mathematical concepts and skills into comprehensive and integrated encounters with natural and social events (Gadanidis et al. 2016), bridging the divide between school mathematics and students' real-world issues. It brings mathematics closer to the students' emotions and allows them to appreciate its beauty.

The Foundations of the Aesthetic Approach in Mathematics Textbooks

When establishing curricula and producing mathematics books, a group of scholars cited Mohamed (2008), Sinclair (2009), Yunus (2012), and Hartono (2016).

Foundations and principles can be summarised as follows:

(1) Unifying the science structure:

The aesthetic approach confirms that mathematics education encompasses the primary components of science, which are the results, which include concepts and theories, the second component of science processes, which are the practices and activities used to arrive at mathematical theories and concepts, such as inference and communication, and finally, the values and ethics that should characterize the students' knowledge as honesty and objectivity, which can be accomplished by presenting paragraphs of text.

(2) Emphasis on the major concepts

The consistency of mathematics curricula around central concepts and their development in conjunction with the development of the "student's cognitive structure, the unity of the mathematics structure and knowledge of the facts and concepts beneath it, and provides more space for aesthetic considerations of mathematical experiences and concepts, as well as emphasizes the importance of mathematics and its applications in their lives.

(3) Presenting Integrated Mathematics Topics

Reduces barriers between scientific disciplines by emphasizing the interconnectivity and interdependence of scientific areas. The aesthetic approach to mathematics education unifies school mathematics curricula across the various fields of mathematics as a science of numbers and measurements. The aesthetic approach enables mathematics to be presented in conjunction with other disciplines and subjects, focusing on fundamental concepts and ideas, seeing them as interconnected concerns, and comprehending them thoroughly. For instance, circumference mathematics can be applied to estimating the circumference of a volcano's crater.

Presenting mathematics integratively enables students to appreciate the beauty of concepts and information, their interconnectedness, the reliance of each domain on mathematics, and the reliance of numerous scientific areas on mathematics, all of which increase motivation to learn.

(4) Linking Mathematics with the Art*s*

It stimulates pupils' imaginations, fosters their creativity, and allows them to express their emotions. The relationship between mathematics themes and concepts and other arts such as architecture, Arabic calligraphy, and others can be defined as the relationship between geometric shapes and decorating art.

(5) Taking care of the Emotional Aspects

It emphasizes taking care of the student's emotional needs without sacrificing cognitive and skill development and arousing students' enthusiasm and curiosity through methods that simplify the mathematics, such as using interesting titles for mathematical topics that motivate students to investigate their content, and providing a learning environment in which the student feels safe, comfortable, and enjoys learning mathematical concepts by connecting them to their interests. The narrative contributes to the excitation of the'students' conscience through its artistic aspects, which pleasure students, acknowledge the efforts of scientists, and help them develop favorable attitudes toward mathematics.

(6) Taking into Account the Students' Mental Level

Because a person perceives beauty in their surroundings based on subjective factors such as mental abilities, it is necessary to consider the gradual introduction of mathematical concepts and standards of mathematical beauty according to the students' mental level, from the tangible to the abstract, when designing the learning experience and situations.

(7) Highlighting the Mathematical Beauty Criteria
It develops a sense of beauty by focusing on two aspects: synthesis, logic, balance, order, and symmetry, as in the concept of an equilateral triangle. This helps students acquire basic knowledge about beauty, which helps them develop their ability to perceive beauty in mathematics topics and enjoy math learning.

Previous Studies

- Previous research has examined the aesthetic approach to mathematics: Obaidah's (2013) study examined the actuality of incorporating mathematics components into teaching from the 'teachers' perspective.
- Abdul Hadi (2014) honed his innovative and mathematical thinking abilities using an artistic approach to mathematics.
- Cellucci (2015) examined the usefulness of mathematics beauty in assisting students' comprehension and discovery, concluding that mathematics beauty aided in boosting students' mathematical discovery and cognitive understanding.
- Satyam (2016) demonstrated the importance of mathematical beauty in achieving surprise and eradicating monotony in educational settings.
- Abdullah's (2019) study demonstrated the efficacy of an aesthetic approach in mathematics instruction to develop humor and aesthetic taste. Both the funny sense scale and the aesthetic taste test revealed statistically significant differences in favor of the experimental group.

It is clear that the previous studies used the aesthetic approach to evaluate teacher performance and teaching using several variables such as mathematical and innovative thinking, sense of humor, the emotion of surprise, understanding, and discovery. In contrast, this study aimed to determine the level of availability of the aesthetic approach in mathematics books.

Methodology and Procedures

Methodology

The researchers used the descriptive-analytical method.

The Study Sample

The study sample consists of primary school mathematics books for the upper grades "fourth, fifth, and sixth" in the first and second semesters in Saudi Arabia for the academic year 1443 AH, a total of six books, and the study used the comprehensive inventory method to the community, due to the study's potential for application to the entire community.

The Study Tool

Following the aesthetic approach's foundations, the researchers collected study data using the content analysis card. By establishing seven major areas of the aesthetic approach, the researchers translated the aesthetic approach's foundations into a content analysis card; the tool's first form had (29) indications distributed throughout seven major domains.
The tool was verified by:

(1) The study tool validity
Its validity is predicated on the opinions of expert arbitrators. The researchers presented the content analysis card in its initial form to a panel of specialized arbitrators, totaling eleven (11) arbitrators, to ascertain their observations and suggestions regarding the analysis card's validity. They provided insightful suggestions and enhancements to the Analysis Card tool that was employed.
As a result, the researchers made appropriate revisions and adjusted the items' linguistic language. Thus, the tool's final version includes (27) indicators distributed among seven primary domains.

(2) The study tool stability
The study employed the re-analysis technique, with a time delay of one year between the first and second analyses (three weeks). The researchers examined a unit of study from the sixth grade of primary school's second semester (the Fractions unit), and then used Holsty's equation to determine the stability. The stability coefficient was (0.93), which is a high number.

Analytical Controls

After confirming the correctness and stability of the content analysis card, the researchers utilized the instrument as follows:

- The analysis's objective was to ascertain the extent to which the aesthetic approach is available in mathematics books for the upper grades at the elementary stage in Saudi Arabia.
- Sample: encompasses all themes covered in elementary school mathematics textbooks for pupils in the upper grades "sixth, fifth, and fourth" throughout the first and second semesters, while taking into account the following procedures:
- Analyzing all book contents except the cover, book introduction, indexes, mid-semester test, chapter test, and cumulative tests.
- Analysis of the artistic approach and its indicators in light of the foundations' list.

- The study includes all scheduled classes and preparation materials and the graphics, shapes, images, and activities included in the content.
- The lesson, its primary activity, exercises, and sub-items were considered repetitions because they all contain the same concept.
- Analyses Categories denoted by the aesthetic approach's foundational domains.
- Categorize partial analyses using the unique indicators indicated in the aesthetic approach areas.
- Unit analysis: Selecting an idea as a unit based on its relation to the study's nature and objectives.

Statistical Treatment

The researchers used:

- Frequencies and percentages.
- Holste's equation calculates the tool stability through the agreement coefficient between the first and second analyses.
- Judgment criterion on the availability level of the aesthetic approach among the study sample according to the Table 1.

Table 1. *Judgment Criterion on the Availability Level of the Aesthetic Approach*

Percentage		**Availability level**
From	**to**	
0%	**20%**	Very low
More than 20%	**40%**	Low
More than 40%	**60%**	Medium
More than 60%	**80%**	High
More than 80%	**100%**	very high

Study Results and Discussion

The Answer to the First Question

"What are the foundations of the aesthetic approach that is required to be available in mathematics textbooks for the higher grades at the primary stage in Saudi Arabia?"

The researcher studied educational literature and existing study on aesthetics. The researchers used aesthetic approach foundations to establish the aesthetic approach's availability in mathematics textbooks and then recast the sub-indicators and analysis in that light, presenting them to a panel of specialized arbitrators and amending them based on their observations. The researchers identified (27) sub-indicators, which included the following:

Unifying the science structure:

(1) The book presents scientists' practices and activities to access mathematical theories and concepts.
(2) The book highlights the scientific values and ethics that accompany the scientific activity.
(3) The book presents the mathematical concepts and findings of scientists.

Emphasis on the major concepts:

(1) The book provides mathematical content according to the cumulative structure characteristic and the tribal requirements for studying different topics.
(2) The book organizes the mathematical content lessons around total concepts that include sub-mathematics concepts.
(3) The book presents the basic conceptual and procedural knowledge in the balanced lesson.

Presenting mathematics in an integrated way:

(1) The book links mathematical applications with other social and scientific disciplines.
(2) The book includes applications that integrate the branches of mathematics (numbers and their operations, geometry, algebra and analysis, statistics, and probability).
(3) The book motivates students to recall previous experiences from different disciplines.
(4) The book includes open activities where students present their relevant experiences in the lesson.

Linking mathematics with arts:

(1) The book links mathematical concepts and arts.
(2) The book refers to applying mathematical concepts in different arts in local and global cultures.
(3) The book provides an opportunity to practice technical activities related to mathematics.
(4) The book encourages the observation and inference of mathematics concepts in the artwork.

Taking care of the students' emotional aspects:

(1) The book provides interesting titles for students for mathematics lessons.
(2) The book includes paragraphs that develop students' positive attitudes towards mathematics, highlighting its functional beauty in their lives.

(3) The book promotes meditation on mathematical concepts and savoring their beauty.
(4) The book urges appreciation and pride in the efforts of mathematicians.
(5) The book presents stories of the mathematical concepts, historical development, and the challenges scientists faced upon discovery.

Taking into account the students' mental level:

(1) Gradual provision of mathematical content from the concrete to the abstract level.
(2) The book presents the topic to consider the individual differences among students.
(3) The book links mathematics to the students' reality.
(4) The book presents the information to stimulate the learner's mental processes.
(5) The book includes various methods of learning assessment.

Mathematics beauty criteria:

(1) The book presents paragraphs explaining the standards of mathematics beauty.
(2) The book is directed to note the standards of mathematical beauty in the mathematical situation.
(3) The book stimulates the observation and conclusion of the mathematical beauty criteria in the students' environment.

The Answer to the Second Question

"What is the availability level of the necessary aesthetic approach foundations in mathematics textbooks for the higher grades at the primary stage in Saudi Arabia?"

The frequencies and percentages of the evidence for each indicator of the analysis card areas in each class book. The researchers present a summary of the results in the light of the aesthetic approach in Table 2.

Table 2. *Summary of the Mathematics Books Results from Analysis for The Fourth-Grade at Primary School in the Light of the Aesthetic Approach*

No.	Domain	Semester	Average Frequency of Domain	Percentage of Domain	Average Percentage of Two Classes	Availability Level
1	Unification of the science structure	First	52	44%	48.5%	medium level
		second	67	53%		
2	Emphasis on major concepts	First	99	85%	72.5%	high level
		second	76	60%		
3	Introducing mathematics in an integrated manner	First	96	49%	35%	low level
		second	35	21%		
4	Connecting mathematics with the arts	First	10	7%	8.5%	very low level
		second	16	10%		
5	Emotional care	First	2	1%	4.5%	very low level
		second	16	8%		
6	Considering the mental level	First	126	65%	60%	medium level
		second	116	55%		
7	mathematics beauty standards	First	2	2%	2,5%	very low level
		second	4	3%		
The average of the aesthetic approach foundations in the fourth-grade books of primary school					33.07%	low level

It is clear from Table 2 the frequencies of the aesthetic approach indicators in mathematics books for the fourth grade at primary school and their percentages concerning some lessons in the book where the percentage of all domains was (33.07%), and the domain (emphasis on major concepts) came first with a percentage of (72.5%). Then, in the second place, the domain (observing the mental level) was with a percentage of (60%). The domain (unifying the science structure) ranked third with a percentage of (48.5%). The domain (presenting mathematics integrated) ranked fourth with a percentage of (35%). Then the domain (linking mathematics with the arts) ranked fifth with a percentage of (8.5%) and the domain (Caring for Emotional Aspects) ranked before the last with a percentage (4.5%). The domain of (Athletic Beauty Standards) ranked last with a percentage of (2.5%).

Table 3. *Summary of the Mathematics Books Results from Analysis for the Fifth-Grade at Primary School According to the Aesthetic Approach*

No.	Domain	Semester	Average Frequency of Domain	Percentage of Domain	Average Percentage of Two Classes	Availability Level
1	Unification of the science structure	First	69	55%	52.5%	medium level
		second	62	50%		
2	Emphasis on major concepts	First	93	74%	73.5%	high level
		second	90	73%		
3	Introducing mathematics in an integrated manner	First	75	45%	38.5%	low level
		second	52	32%		
4	Connecting mathematics with the arts	First	7	4%	6.5%	very low level
		second	15	9%		
5	Emotional care	First	3	2%	3%	very low level
		second	7	4%		
6	Considering the mental level	First	123	58%	56.5%	medium level
		Second	113	55%		
7	mathematics beauty standards	First	0	None	1%	very low level
		second	3	2%		
The average of the aesthetic approach foundations in the fifth-grade books of primary school					33.07%	low level

It is clear from Table 3 the frequencies of the aesthetic entrance indicators in mathematics books for the fifth grade at the primary school, and their percentages concerning some lessons in the book where the percentage of all domains reached (33.07%), and the domain (emphasis on major concepts) came first with a percentage of (73.5%). Then, in the second place, the domain of (observing the mental level) was with a percentage of (56.5%). The domain (unifying the science structure) came in third place, with a percentage of (52.5%). The domain (presenting mathematics in an integrated manner) ranked fourth, with a percentage of (38.5%). Then the domain (linking mathematics with the arts) ranked fifth with a percentage of (6.5%). The domain (Caring for Emotional Aspects) ranked penultimate was with a percentage (3%). The domain of (Mathematics Beauty Standards) ranked last with a percentage (1%).

Table 4. *Summary of the Mathematics Books Results from Analysis for the Sixth-Grade at Primary School According to the Aesthetic Approach*

No.	Domain	Semester	Average Frequency of Domain	Percentage of Domain	Average Percentage of Two Classes	Availability Level
1	Unification of the science structure	First	59	56%	57.5%	medium level
		second	55	59%		
2	Emphasis on major concepts	First	91	87%	84.5%	high level
		second	77	82%		
3	Introducing mathematics in an integrated manner	First	43	31%	32.5%	low level
		second	43	34%		
4	Connecting mathematics with the arts	First	20	14%	17%	very low level
		second	25	20%		
5	Emotional care	First	22	11%	11.5%	very low level
		second	22	12%		
6	Considering the mental level	First	105	53%	56%	medium level
		Second	91	59%		
7	mathematics beauty standards	First	5	5%	2.5%	very low level
		second	0	None		
The average of the aesthetic approach foundations in the sixth-grade books of primary school					37.3%	low level

It is clear from Table 4 the frequencies of the aesthetic approach indicators in mathematics books for the sixth grade at primary school, and their percentages concerning some lessons in the book where the percentage of all domains reached (33.07%), and the domain (emphasis on major concepts) came first with a percentage of (84.5%). Then, in the second place, the domain of (unifying the science structure) was with a percentage of (57.5%) and ranked third in the domain of (observing the mental level) with a percentage (56%). The domain (providing mathematics in an integrated manner) ranked fourth, with a percentage of (32.5%). Then the domain (linking mathematics with the arts) ranked fifth with a percentage (17%). The domain (Caring for Emotional Aspects) ranked was with a percentage of (11.5%). The domain of (Athletic Beauty Standards) ranked last with a percentage of (2.5%).

Table 5. *Summary of the Mathematics Books Results from Analysis for the Higher Grades at the Primary Stage According to the Aesthetic Approach*

No.	Domain	Percentage of the fourth-grade books	Percentage of the fifth-grade books	Percentage of the sixth-grade books	Average percentage of higher grades books	Availability Level
1	Unification of the science structure	48.5%	52.5%	57.5%	52.8%	medium level
2	Emphasis on major concepts	72.5%	73.5%	84.5%	76.8%	high level
3	Introducing mathematics in an integrated manner	35%	38.5%	32.5%	35.3%	low level
4	Connecting mathematics with the arts	8.5%	6.5%	17%	10.6%	very low level
5	Emotional care	4.5%	3%	11.5%	6.3%	very low level
6	Considering the mental level	60%	56,5%	56%	57.5%	medium level
7	mathematics beauty standards	2.5%	1%	2.5%	2%	very low level
The average of the aesthetic approach foundations in the higher grades books of the primary school		33.07%	33.07%	36.8%	34.47%	low level

As shown in Table 5, the results of analyzing the content of mathematics books for the higher grades at the primary stage, in general, demonstrated consistency in terms of the convergence of percentages for the availability of the aesthetic approach in mathematics books for each domain of the aesthetic approach between each grade of the higher grades at the primary stage. This consistency can be attributed to the spiral organization of the current (emphasizing the major concepts, taking into account the mental level, unifying the science structure, presenting mathematics in an integrated manner, linking mathematics with the arts, taking care of emotional aspects, standards of mathematical beauty).

These results can be discussed as follows:

The domain of emphasizing major concepts gained the top spot with 76.8 percent and a high degree of availability, owing to the spiral organization of the mathematics curriculum. The major concepts are introduced in broad strokes first, followed by their comprehensive presentation at each level. This finding is consistent with Al-Maliki and Al-Riyashi (2019) and Khalil and Al-Suloli (2015) about the order in which concepts are presented, and lessons are connected. However, there are certain inadequacies in the indication of conceptual and procedural knowledge due to publications focused exclusively on the procedural element. Concentrating on the procedural part of books affects the domain's availability in books, which is consistent with (Al-Ahmadi 2020).

With a percentage of 57.5 percent and availability at an average degree, the realm of considering pupils' mental abilities came in second place. For the age stage, and by presenting information in ways that do not adequately account for the individual variances amongst pupils, the presentation does not stimulate the student's mental processes. This finding is consistent with (Al-Ahmadi 2020) and (Khalil and Al-Suloli 2015) findings that the Gradual presentation of mathematical concepts and ideas from the tangible to the abstract is missing in illustrations, representations, and real-world situations.

The domain of unifying the science structure came in third place with a percentage of (52.8 percent) and a medium degree of availability. This result stems from the fact that mathematics textbooks presented the science structure in various ways, initially emphasizing the cognitive aspect and then practicing. As a result, scientists' activities decreased by 49%, while scientific values and ethics decreased by 10%, and this result is consistent with (Al-Tamimi 2017).

The domain of integrated mathematics came in fourth place with a percentage of 35.4 percent and a low degree of availability. This finding is consistent with (Khalil and Al-Suloli 2015) regarding the lack of strong connections between mathematical concepts and other fields. However, the NCTM's School Mathematics Principles (NCTM 2000) emphasized the importance of mathematics in students' scientific and practical lives.

- The domain of mathematics and the arts was ranked fifth, with a proportion of 10.7 percent and a very low degree of availability. This is due to a lack of interest in the content or activities that support the practice of mathematically related artistic activities and a lack of presentation of content for applications of mathematics in the arts that benefit students. This finding concurs with (Khalil and Al-Sulouli 2015) regarding the book's inclusion of activities.

- The care of emotional aspects domain was ranked sixth with a percentage of 4.9 percent and availability at a very low level, indicating the content's inability to pique students' enthusiasm and motivate them to learn mathematics through methods that simplify the subject rather than as an abstract cognitive subject. This finding is compatible with (Al-Maliki and Al-Ghamdi 2019), who assert that mathematics books have weaknesses, including the stories of historical scientists and their endeavors to build mathematics and activities with emotional purposes.

- While the domain of Mathematical Beauty criterion attained the seventh and final position with a percentage of 2.2 percent and availability at a very low level, this is obvious by the absence of paragraphs explaining the Mathematical Beauty criteria in mathematics books. Mathematics incorporates aesthetic values, which may result from the emphasis on cognitive and skill development while building mathematics curricula; however, other research has stressed the aesthetic dimension's importance in mathematics (Sinclair 2009).

Recommendations

Due to the spiral organization of the mathematics curriculum, the domain of emphasising major concepts took the top rank with 76.8 percent and a high degree of availability. The major concepts are introduced in broad strokes first, followed by a detailed discussion of each subject at each level. This finding is consistent with Al-Maliki and Al-Riyashi (2019) and Khalil and Al-Suloli (2015) about the sequencing concepts and lessons. However, there are some shortcomings in the indication of conceptual and procedural knowledge due to publications that are purely procedural in nature. Concentrating on the procedural portion of books affects the availability of the domain in books, which is compatible with Al-Ahmadi (2020).

The domain of considering pupils' mental talents came in second place with a percentage of 57.5 percent and availability at an average level. Due to the students' age stage and the fact that the information is presented in ways that do not appropriately account for individual differences among pupils, the presentation does not contribute to stimulating the students' mental processes. This study is consistent with Al-Ahmadi (2020) and Khalil and Al-Suloli (2015) findings that depictions, representations, and real-world scenarios lack a gradual presentation of mathematical concepts and ideas from the tangible to the abstract.

With a proportion of 52.8 percent and a moderate degree of availability, the domain of unifying the science framework is placed in third place. This outcome is because mathematics textbooks presented the science framework in various ways, emphasizing first the cognitive part and later the practical aspect. As a result, scientists' activities declined by 49%, while scientific values and ethics declined by 10%, a finding congruent with Al-Tamimi (2017).

With a proportion of 35.4 percent and a low degree of availability, the domain of integrated mathematics is placed in fourth place. This study corroborates (Khalil and Al-Suloli 2015) the absence of strong linkages between mathematical notions and other domains. However, the NCTM's School Mathematics Principles (NCTM 2000) emphasized mathematics' role in students' scientific and practical lives.

- Mathematics and the arts were ranked fifth, with a share of 10.7% and a very low degree of availability. This is due to a lack of interest in the content or activities that promote mathematically linked creative activities and a dearth of content relating to mathematical applications in the arts that benefit students. This conclusion is consistent with Khalil and Al-Sulouli (2015) about the inclusion of activities in the book.

- The care of emotional aspects domain was ranked sixth with a percentage of 4.9 percent and availability at an extremely low level, indicating the content's inability to pique students' enthusiasm and motivate them to learn mathematics through simplified methods rather than as an abstract cognitive subject. This finding is consistent with Al-Maliki and Al-Ghamdi (2019) who say that mathematics books have flaws, including historical scientists' stories and their efforts to develop mathematics and activities for emotional reasons.

- While the domain of Mathematical Beauty criterion was ranked seventh and final with a percentage of 2.2 percent and availability at a very low level, this is evident by the absence of paragraphs explaining the Mathematical Beauty criteria in mathematics books. Mathematics integrates aesthetic values, resulting from the emphasis placed on cognitive and skill development while developing mathematics curricula, but other studies have emphasized the importance of the aesthetic dimension in mathematics (Sinclair 2009).

Suggestions

The researchers suggest the following studies:

(1) Evaluating and developing mathematics curricula according to the aesthetic approach in general education stages.
(2) Comparative studies to analyze mathematics textbooks for the primary and intermediate levels in Saudi Arabia and developed countries with the TIMSS results.
(3) Evaluating the teaching performance of mathematics teachers in general education stages according to the aesthetic approach.

References

Abdul Hadi AMR (2014) A program based on the aesthetic approach in mathematics to develop innovative thinking and mathematical thinking skills among basic education students. *Reading and knowledge Journal* (149): 61–87.

Abdullah AMG (2019) Using the aesthetic approach in teaching mathematics to develop humor and aesthetic taste for fifth-grade students. *Mathematics Education Journal* 22(2): 80–144.

Abu Zaid AMAH, Selim MS, Moawad LIA (2017) The effectiveness of the aesthetic approach in teaching biology to develop some scientific concepts and the students' and teachers' opinions at the secondary stage towards its use. *Studies in University Education: Ain Shams University – Education Faculty - University Education Development Cente*r 35: 598–610.

Ahmed SSAH (2018) Mathematics with aesthetic prospects. Paper Presented at *The Sixteenth Annual Scientific Conference: Developing Mathematics Teaching and Learning to achieve a Quality Culture.* Egyptian Association for Mathematics Education.

Al-Ahmadi SMS (2020) A comparative study of the fractions content in mathematics textbooks in Saudi Arabia and Singapore. *Egyptian Association for Mathematics Education* 23(2): 231–283.

Al-Maliki AMS, Al-Riyashi HAH (2019) *Evaluating the mathematics curriculum content in the higher grades at the primary stage according to the mathematical proficiency components*. Unpublished Master's Thesis. Abha: King Khalid University.

Al-Saadowi AS (2010) *A teacher's guide to performance-based assessment from theory to practice*. Arab Bureau of Education for the Gulf States.

Al-Sherbiny F (2005) A*esthetic education in educational curricula to confront contemporary issues and problems*. Cairo: Book Center for Publishing.

Al-Tamimi ARIA (2017) The extent to which scientific values are included in mathematics textbooks in higher grades at the primary stage: an analytical study. *Tanta University - Education Faculty* 68(4): 504–529.

Asiri M, Al-Omrani HM, Al-Thakir FA (2013) *School Mathematics Principles and Standards: National Council of United States Mathematics Teachers*. Arab Office of Education for the Gulf States.

Cellucci C (2015) Mathematical beauty, understanding, and discovery. *Foundations of Science* 20(4): 339–355.

Center for Research Excellence in the Development of Science and Mathematics Education (2015) *Evaluation study for the project to develop mathematics and natural sciences in public education in Saudi Arabia (extended study summary)*. Riyadh: Center for Research Excellence in Developing Science and Mathematics Education at King Saud University.

Gadanidis G, Borba M, Hughes J, Lacerda HD (2016) Designing aesthetic experiences for young mathematicians: a model for mathematics education reform. *International Journal for Research in Mathematics Education* 6(2): 225–244.

Girod M, Twyman T, Wojcikiewicz S (2010) Teaching and learning science for transformative, aesthetic experience. *Science Teacher Education Journal* 21: 801–824.

Hartono T (2016) Aesthetics in school mathematics: a potential model and a possible lesson. *The Mathematics Enthusiast* 13(3).

Khalil IA, Al-Suloli MS (2015) The inclusion level after the operations dimension of mathematical power in the mathematics books of the higher classes at the primary stage in Saudi Arabia. *Resalet Al-Khalej Alaraby* 147: 55–71.

Mhenni NSM (2018) The effect of using the aesthetic approach in teaching history to develop dimensions of taste and some values among basic education students. *The Educational Society for Social Studies Journal* 101: 244–259.

Minara Y (2017) Aesthetic dimension in mathematics at school level. *International Journal of Multidisciplinary Research and Development UGC* 4(6): 493–496

Ministry of Education (2019) *Education and Saudi vision 2030.* Available at: https://www.moe.gov.sa/ar/Pages/vision2030.aspx.

Mohamed JH (2008) *Develop an aesthetic taste for mathematics*. Culture for Development Association and Suhag University.

Obaidah NEAH (2013) Evaluating the reality of employing the mathematics components in teaching from the secondary school teachers' point of view. *Educational Journal: Kuwait University - Scientific Publication Council* 27: 375–425.

Saleh MAH, Matalqa AMA (2018) *Teaching Islamic education in the light of the aesthetic approach and its effectiveness in forming value-building, developing cognitive achievement, and moving towards studying it*. Unpublished Doctoral Thesis. Yarmouk University.

Satyam VR (2016) The importance of surprise in mathematical beauty. *Journal of Humanistic Mathematics* 6(3): 232–248.

Sinclair N (2009) Aesthetics as a liberating force in mathematics education. *ZDM Mathematics Education* 41: 45–60.

Suleiman KRK (2016) A proposed teaching model based on the aesthetic approach to developing physical concepts and reflective thinking skills for first-year secondary students. *Reading and Knowledge Magazine: Ain Shams University – Education Faculty - Egyptian Society for Reading and Knowledge* 182: 93–130.

Yunus IMMM (2012) *A proposed curriculum in science for the preparatory stage in light of the aesthetic approach and its effectiveness in developing cognitive achievement, values, and the trend towards the study of science*. Unpublished Doctoral Thesis. Cairo: Ain Shams University.