Sets, Properties and Truth Values: A Category-Theoretic Approach to Zermelo's Axiom of Separation

*By Ivonne Pallares Vega**

In 1908 the German mathematician Ernst Zermelo gave an axiomatization of the concept of set. His axioms remain at the core of what became to be known as Zermelo-Fraenkel set theory. There were two axioms that received diverse criticisms at the time: the axiom of choice and the axiom of separation. This paper centers around one question this latter axiom raised. The main purpose is to show how this question might be solved with the aid of another, more recent mathematical theory of sets which, like Zermelo's, has numerous philosophical underpinnings.

Keywords: *properties of sets, foundations of mathematics, axiom of separation, subobject classifier, truth values*

Introduction

Set theory is a field of mathematics that has the peculiar property of allowing the "codification" of many mathematical concepts such as those of number, cartesian product, function, group, ring and many others. Another branch of mathematics that also has this property is category theory, although its "encoding" is quite different from that of set theory. It is mostly this feature shared by both theories that makes them quite amenable to philosophical investigations. Indeed, the word most commonly used is not "codification" or "encoding" but *foundation*, although this latter term tends to have different meanings for different philosophers of mathematics. In 1964 the American mathematician F. William Lawvere published the first axiomatization of the concept of *set* within the framework of category theory[1]. In this paper I shall refer to the more fully developed version he published jointly with Robert Rosebrugh in 2003[2]. For Zermelo´s axiomatization I will base my discussion on a paper entitled "Investigations in the foundations of set theory I", which he published in 1908[3]. The discussion is centered around one axiom, one that deals with relation between certain properties and the existence of subsets of a given set. Zermelo calls it the *Axiom of separation* and *LR* call it *Membership representation via truth values*[4].

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[1]Reprinted in Lawvere (2005).

[2]Lawvere and Rosebrugh (2003). I shall refer to this axiomatization throughout the text as *LR*.

[3]Zermelo E (1908a). Throughout the main text I will refer to this paper as *Zermelo 1908a*, and to Zermelo (1908) as *Zermelo 1908*.

[4]Within the more general context of topos theory (an important branch of category theory), this is called the Subobject Classifier axiom.

https://doi.org/10.30958/ajphil.1-3-2 doi=10.30958/ajphil.1-3-2

When philosophers of mathematics discuss foundational issues concerning set theory, they usually have in mind what has become to be known as "Zermelo-Fraenkel set theory" or "Zermelo-Fraenkel set theory with the axiom of choice" (abbreviated as ZF and ZFC, respectively). Ever since category theory entered the scene in discussions of foundational issues, ZF (and *per force* Zermelo's own axiomatization) has been discredited to a large extent[5]. I believe that the categorical approach to set theory still owes something of value to *Zermelo 1908a*, even if only for reacting against it on various fronts, some of which I will present in the section concerning the concept of 'function'[6]. So by basing my exposition on *Zermelo 1908a*, rather than on current versions of set theory like ZF, I intend this paper to be partly a tribute to Zermelo. Indeed, both *Zermelo 1908* and *Zermelo 1908a* include the axiom of choice, and so does the categorical approach. One of my purposes here is to show the sense in which the categorical approach includes part of Zermelo´s axiom of separation. But my main purpose, and now as a tribute to the categorical approach, is to show how the categorical version of this axiom solves a problem raised by its original formulation. Thus rather than discrediting ZF in favor of the categorical approach, I propose here to see the latter as an instance of conceptual progress within mathematics.

Both theories are rich in consequences, not just mathematical but also philosophical. However, I shall only present those axioms, concepts, theorems and definitions from each theory that are necessary for the exposition or useful for illustrating purposes, thus keeping technical details to a minimum.

Zermelo's Set Theory and Properties of Sets

According to *Zermelo 1908a*, set theory is about a certain domain of *individuals* or *objects*, among which are the *sets*. One of the functions of the axioms is precisely to characterize the latter ones. In the introductory part he says that Cantor's 1895 original definition of a set as "any collection into a whole *M* of definite and separate objects *m* of our intuition or our thought" (Cantor 1955, p. 85) needs to be restricted, for, as it was well known, it leads to certain contradictions such as the Russell paradox concerning the set of all sets not belonging to themselves. Thus the axioms are meant to serve the function of restricting the notion of 'set' so as to prevent the theory to give rise to paradoxes such as Russell's[7].

[5]Ernst (2017, p. 69), e.g., talks of categorical foundations replacing set theory.

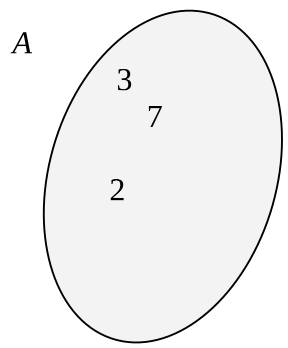
[6]Lawvere (2005) describes its own contents as "a way out of the [Zermelo-Fraenkel] impasse" (p. 6) where the latter's membership-theoretic definitions and results are "for a beginner bizarre" (ibid.) Almost forty years later the claim is that "[…] from the ongoing investigation of the ideas of sets and mappings, one can extract a few statements called *axioms* […] The use of this axiomatic method makes naive set theory rigorous and helps students to master the ideas without superstition." (*LR*, pp. ix–x, emphasis in the original). Be this as it may, I am nonetheless convinced that there is much more of value in category theory, especially for all those interested in philosophy of mathematics.

[7]See, e.g., Fraenkel et al. (1973, p. 18). However and in contrast, Gregory H. Moore has argued extensively in favor of the claim that Zermelo's main motivation for axiomatizing set theory was instead that of defending his proof of the well-ordering theorem, which he first gave in 1904 and

Set theory is based on the notion of *membership*, which is taken as basic, not definable in terms of any other more basic concepts. From the seven axioms in *Zermelo 1908a*, five of them are about the formation of new sets from other sets previously given or assumed to exist. The exceptions to these are the axiom of the empty set (which postulates the existence of a set with no elements) and the axiom of extensionality. Except for this axiom, all the remaining axioms that postulate the existence of certain sets start from *sets* already given, not from general "objects" of what Zermelo calls the "domain" of set theory, that is, not from objects that may not be sets in the sense of the other axioms. Following most introductory textbooks on set theory, I will ignore this point and consider the universe of the theory as consisting of only sets whose elements are themselves sets, whose elements are also sets, and so forth.

The symbol for the relation of membership between sets is $\in$. There are several ways for denoting a set. Suppose that A is a set whose elements or members are the natural numbers from 1 to 4. We may then write $A = \{1, 2, 3, 4\}$. Or, if N denotes the set of *all* natural numbers, we may also write $A = \{x \in N \mid 1 \leq x \leq 4\}$, where the symbol '|' is read as "such that" or "with the property that". This notation is useful, for example, when one cannot list all the elements of a set, either because they are too many for it to be feasible in practice or because the set is infinite. The set of even numbers, for instance, is infinite, so we can denote it simply by writing $\{x \in N \mid x \text{ is even}\}$. Now, in order to express that, for example, a certain number is *not* an element of A, we use the symbol '$\notin$'. So for instance, we have that $5 \notin A$.

Sometimes it is useful to picture sets by means of what are called Venn diagrams. For example, if $A = \{2, 3, 7\}$ we may draw the following diagram



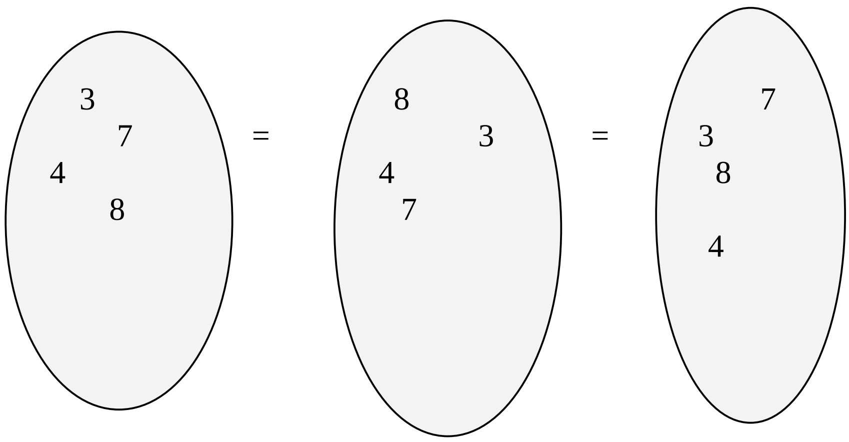
Zermelo's first axiom gives us a criterion for determining whether or not any two given sets are equal.

then later modified in Zermelo (1908). See, for instance, Moore (2013), especially Chapter 3. I tend to favor Moore's view, if only because it makes Zermelo's efforts to prove the well-ordering theorem—which of course include his axiomatization of set theory—more interesting from a mathematical, philosophical and historical perspective.

Axiom I (Axiom of Extensionality)

Given any two sets M and N, if every element of M is also an element of N and vice versa, then $M = N$.

One may see this axiom as saying that a set is completely determined by its elements or, equivalently, as saying that, for any two given sets, in order for them to be different, there *must be* at least one element belonging to one of them but no to the other set. One consequence of this axiom is that, for example, the sets {3, 7, 4, 8}, {8, 3, 4, 7} and {7, 3, 8, 4} are all one and the same set. Venn diagrams may illustrate this more clearly[8]:



So the order in which the elements of a set are listed does not affect its identity. One might say that what gives a set its "substance" are its elements, regardless of the order in which one thinks of them. The categorical approach to sets contrasts with this way of thinking about sets; for within category theory, the focus is not so much on what the elements of a set are but on how it relates to other sets.

In the next section we will see how one can capture the idea of *order* in terms of the membership relation. The remaining three axioms I will be considering are of a different nature than the axiom of extensionality since they all postulate the existence of certain sets.

Axiom II (Axiom of Elementary Sets)

There exists a set that has no elements at all. If a is a set, there exists a set $\{a\}$ containing a as its only element. If a and b are sets, then there always exists a set $\{a, b\}$ whose only elements are the sets a and b.

One question immediately arises: How many sets with no elements are there? If we suppose there are at least two different empty sets, let us call them A and B, then, according to Axiom I, one of them must have at least one element that does

[8]In principle, Venn diagrams can be of any shape and size, although they are most commonly drawn as above or as circles. See e.g., *LR*, p. 1.

not belong to the other (empty) set. But clearly this is not possible, since by hypothesis neither *A* nor *B* have any elements. In this way, we have arrived at our first theorem:

Theorem (Uniqueness of the empty set)
There is only one empty set.

□.

Since there is just one empty set, we are justified in using a special symbol for it. As it is customary, I will use the symbol ∅, although it is sometimes also denoted as { }.

Before presenting the axiom of separation, which is Zermelo's Axiom III, I shall present one more axiom of existence which, together with Axiom II, is useful for illustrating the nature of the set universe as well as the special encoding set theory allows for defining mathematical objects such as the natural numbers. However, I shall retain Zermelo's numbering for his axioms.

Axiom VII (Axiom of Infinity)

There is a set that has the empty set as one of its elements and it is such that whenever a set *a* belongs to it, so does the set $\{a\}$.

With Axiom II at hand, we can prove, for example, the existence of the sets $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$ and $\{\{\varnothing\}\}$. Clearly, none of these three sets is equal to the empty set: they all have either one or two elements. Axiom VII then postulates the existence of the following set

$$\left\{\varnothing, \{\varnothing\},\{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots\right\}$$

and this is a set that can be used to encode the natural numbers. For we simply let the number 0 be the empty set, 1 the set $\{\varnothing\}$, 2 the set $\{\{\varnothing\}\}$, and so on. The usual operations of addition and multiplication can be defined for these sets and in such a way that the elementary properties of arithmetic (such as commutativity and associativity) hold. Moreover, equations such as 1+1 = 2 are theorems of the theory, that is to say, they can be proved within the theory (after all, they only involve the relation of equality between sets which in turn is given in terms of the membership relation).

There are, however, other ways for encoding the set of natural numbers. Consider, for example, the following

$$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\{\varnothing, \{\varnothing\}\}, \ldots\}$$

In this case, we let $0 = \varnothing$, $1 = \{\varnothing\}$, $2 = \{\varnothing, \{\varnothing\}\}$, $3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing, \{\varnothing\}\}$, etc. Here too, one can define the operations of addition and multiplication and prove the usual arithmetical properties of them.

It is tempting to see either definition of the natural numbers as providing an answer to the question of what (natural) numbers are[9]. There are, however, strange well-known consequences one can draw from either one. For example, in both cases it is true that $0 \in 1 \in 2 \in 3 \in \ldots$ And assertions such as these simply do not make sense outside the context of the set-theoretic encoding of numbers, neither are they taught as the basic properties of numbers[10]. This is indeed one of the criticisms that has been raised against set-theoretic foundations of mathematics: they are at odds with mathematical practice[11]. In fact, the categorical approach to sets aims at bringing set theory closer to, and useful for, the actual practice and development of mathematics[12]. But mathematics evolves and so do its practice, problems and tools, as I hope will illustrate the comparison I will draw between Zermelo's axiom of separation and its categorical version.

For stating the last axiom that I shall be considering, we need the following definition.

Definition

Let M and N be given sets. If every element of N is also an element of M, then we say that N is a *subset* or *part* of M, and we denote this by $N \subseteq M$. If $N \subseteq M$ but N is different from M (that is, if there is at least one element of M that does not belong to N), we say that N is a *proper subset* of M, and write $N \subset M$[13].

For example, let $M = \{1,2,4,7,9\}$ and $N = \{2,7\}$. By using Venn diagrams, we can picture that $N \subset M$ as follows

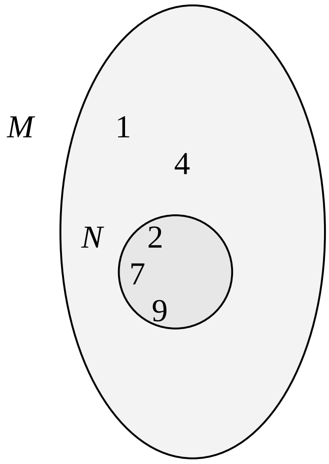
[9]Benacerraf (1965) develops a now classical argument concerning the issue of whether numbers can be sets or not.

[10]However, within set-theoretic foundations, one can make sense of this assertion: for example, for the set $\{\varnothing, \{\varnothing\},\{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots\}$ one can simply *define* or reduce the relation "n is less than m" to the assertion "$n \in m$". In this way one introduces an order among the elements of "the" set of natural numbers.

[11]Leinster (2014), besides giving a concise introduction to the category of sets, also presents in a straightforward manner some of these criticisms.

[12]See, e.g., McLarty (2017), especially pp. 11ff.

[13]The concept of *proper subset* allows us to define a relation of order among the elements of the set $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\{\varnothing, \{\varnothing\}\}, \ldots\}$: we say that $m < n$, if and only if $m \subset n$.



Suppose now that *M* is a non-empty set. Then clearly, for each $x \in M$, *x* is an element of *M*. Hence, $M \subseteq M$. Now, in order for the empty set to *not* be a subset of a given set *M* (whether *M* is empty or not), the empty set must, according to the above definition, have at least one element that does not belong to *M*. But this is clearly not possible: by definition, the empty set does not have any elements. So, we arrive at the following

Theorem
For any given set *M*, $\varnothing \subseteq M$.

□.

As a particular case, we have the following

Corollary
$\varnothing \subseteq \varnothing$.

□.

Thus, *any* given set *M* distinct from the empty set has at least *two different* parts or subsets: *M* itself and the empty set $\varnothing$. And, if *M* is the empty set, then it has just one part: *M* itself. We now come to our last axiom, the axiom of separation or the subset axiom. In *Zermelo 1908* this is phrased as follows:

> All elements of a set *M* that have a property *F* well-defined for every single element are themselves the elements of another set, M_F, a "subset" of *M* (Zermelo 1908, p. 183).

So, this is an axiom linking the process of forming new sets with (well-defined) *properties*[14]. For his second formulation of the axiom, *Zermelo 1908a* introduces the following definition:

[14]In Fraenkel et al. (1973), the authors assert that the axiom of separation "turned out to be inconsistent and therefore cannot be used as an axiom of set theory. However, since this axiom is so close to our intuitive concept of set we shall try to retain a considerable number of [its] instances" (p. 32)—where the instance the authors used for deriving a contradiction was the (non-definite!)

> A question or assertion *F* is said to be *definite* if the fundamental relations of the domain, by means of axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise a "propositional function" *F*(*x*), in which the variable *x* ranges over all individuals of a class *R*, is said to be definite if it is definite for *each single* individual *x* of *R*. Thus, the question whether $a \in b$ or not is always definite, as is the question whether $M \subset N$ or not.[15]

The second formulation of the axiom in *Zermelo 1908a* is then as follows:

Axiom III (Axiom of Separation or Subset Axiom)

Whenever the propositional function *F*(*x*) is definite for all elements of a set *M*, *M* possesses a subset M_F containing as elements precisely those elements *x* of *M* for which *F*(*x*) is true (Zermelo 1908a, p. 201).

Zermelo then observes

> […] In the first place, sets may never be *independently defined* by means of this axiom but must always be *separated* as subsets from sets already given; thus contradictory notions such as "the set of all sets" or "the set of all ordinal numbers" […] are excluded. In the second place, moreover, the defining criterion must always be definite in the sense of our definition […] (that is, for each single element *x* of *M* the fundamental relations of the domain must determine whether it holds or not), with the result that, from our point of view, all criteria such as 'definable by a finite number of words' hence the 'Richard antinomy' […] vanish.[16]

With this axiom, Zermelo had then two aims: on the one hand, to restrict separation to separation within an already given set in order to exclude totalities such as the set of all sets that are not members of themselves (a totality that leads to the well-known Russell's paradox); and, on the other hand, to exclude "illegitimate" properties such as "*x* is a number definable in a finite number of words in natural language" (a property that leads to the Richard's paradox)[17]. Indeed, as a direct application of this axiom, Zermelo proves the following

Theorem

Every set *M* possesses at least one subset M_0 that is not an element of *M*.

Proof. For every element *x* of *M* it is always definite whether $x \in x$ or not (at least the axioms do not exclude the possibility that there may be a set *x* such that $x \in x$). The negation $x \notin x$ is therefore also a definite propositional function. By the axiom of separation, the collection $M_0 = \{x \in M \mid x \notin x\}$ is therefore a set. We have then

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property of not being a member of itself: $x \notin x$. My point is, however, that even those dismissing Zermelo's notion of definiteness, they consider it essential to the concept of set itself that it is tightly connected with *properties* of its elements.

[15]Zermelo (1908a, p. 201), emphasis in the original.

[16]*Loc.cit*., emphases in the original.

[17]See Ebbinghaus' introductory note to Zermelo (1929) in Zermelo (2010, pp. 352–357). Concerning Richard's paradox, see Taylor (1993, pp. 547–549).

that either that $M_0 \in M_0$ or $M_0 \notin M_0$. If $M_0 \in M_0$, then M_0 must satisfy the propositional function $x \notin x$ that characterizes all the elements of M_0 and this means that the assertion $M_0 \notin M_0$ is true. But this contradicts our hypothesis that $M_0 \in M_0$. We conclude then that M_0 is *not* an element of M_0. If, finally, $M_0 \in M$ then, then M_0 would satisfy the condition (given by the propositional function $x \notin x$) characterizing all the elements of M_0 and would thus belong to M_0, a possibility which we have already excluded. Therefore, M_0 is a subset of M such that $M_0 \notin M$. □.

Zermelo then argues that this theorem implies that *not* all objects of the domain of individuals or objects that set theory is about, can be elements of one and the same set. In other words, the theorem implies that this domain is *not* itself a set, so that there is no such thing as the set of all sets. With this result at hand, Zermelo manages to exclude the existence (as a set *of the theory*) of the collection of all sets that are not members of themselves, which is the starting point of the Russell paradox. As Zermelo points out, and except for the case of certain axioms of existence (including the three ones previously given here), "sets […] must always be *separated* as subsets from sets already given"[18], and this separation is to be carried out by means of certain properties.

Besides assertions of the form $x \in y$, $x = y$, $x \notin y$ and $x \neq y$, we may safely conclude from other applications Zermelo gives of the subset axiom, that at least the following are also *definite* propositional functions that we may use in order to form new sets out of previously given sets: if $F(x)$ and $G(x)$ are definite propositions, so are their corresponding negations $\neg F(x)$ and $\neg G(x)$, their conjunction $F(x) \wedge G(x)$ and their disjunction $F(x) \vee G(x)$. So the property of being *definite* is closely connected to logic at a syntactical level. But it is also connected to logic (to *classical* logic, as it turns out) at a semantical level. For according to Zermelo an assertion is definite if "the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether *it holds or not*"[19]. Thus, if $F(x)$ is a definite propositional function and M is a given set then, for each element x of M, the assertion "$F(x) \vee \neg F(x)$" is always true. Or, in other words, if $F(x)$ is a well-defined property of x, then either x has that property F or it does not.

To say of these definite or well-defined properties that the laws of logic determine whether they hold or not, is to say at least that such properties are *bivalent*: if F expresses a property meaningful for all elements x of a given set M then, either the assertion $F(x)$ is true or it is false and never both. In logic, if an assertion $F(x)$ is true (or false, as the case may be) we say of $F(x)$ that its *truth value* is the value *true* (or, accordingly, the truth value *false*).

In Fraenkel AA, Bar-Hillel Y, Levy A (1973) the authors give a succinct account of what they see is the problem with Zermelo's notion of definite

[18]Zermelo (1908a, p. 202), emphasis in the original.
[19]*Loc.cit*., my emphases.

property[20]. They first give the following reformulation of Zermelo's axiom of separation

> For any set *a* and any condition *B*(*x*) on *x* there exists the set that contains just those members of *a* which fulfil the condition *B*(*x*) (Fraenkel et al. 1973, p. 36).

The authors then explain why, in spite of appearances to the contrary, this formulation is different from Zermelo's. In the first place, they say, the notion of a 'condition *B*(*x*) on *x*' is a well-defined notion, for they have previously defined what is called in logic the 'object language' in which these conditions on *x* are to be expressed. This language is, of course, a first-order language with equality and the basic notion of membership. It is a language with precise rules for forming "conditions on *x*" out of the basic membership relation with the aid of the logical connectives and the quantifiers and in such a way that given any finite sequence of symbols of the language, one can always determine whether it is well-formed or not, that is to say, whether it is in fact a condition on *x*[21]. Zermelo did not make a distinction between the object language and what is called the metalanguage (which is the natural language one uses for expressing or explaining axioms, theorems, and so forth). This is one of the reasons why he rejected solutions such as the one proposed by these authors, for the well-formed expressions in the object language are always *finite* sequences of symbols, and Zermelo thought that one of the purposes of set theory was precisely to give an account of the concept of (finite) number (see Zermelo 1929, pp. 359, 363). Thus, the authors continue, since "Zermelo did not have any particular object language in mind […] his notion of a statement *B*(*x*) was quite vague" (Fraenkel et al. 1973, p. 37). Once one is clear about the object language, according to them one can then interpret Zermelo as follows:

> What Zermelo meant by saying that the truth or falsity of [a statement] *B* is determined by the primitive relations of the system is […] that once the primitive relation of the system (namely, the membership relation) is "given" then the very meaning of *B* makes it either true or false. Using modern terminology we can say that *B* is definite if it belongs to a formal system with an interpretation which makes *B* true or false; likewise, *B*(*x*) is definite for a class *R* of objects of the system if *B*(*x*) belongs to an interpreted formal system which makes *B*(*x*) true or false for every member of the class *R*. (*Loc.cit).*

If Zermelo did indeed not make a clear distinction between the object language and the metalanguage, it seems difficult to interpret him as in the above passage. In the final section, I will present Zermelo's own account of the concept of definite property, an account which he gave in 1929 and in response to the criticisms his 1908a version received. Those criticisms amount basically to claiming that his notion of definiteness is imprecise, that it lacks in mathematical rigor. Let it suffice it for the moment to say that the three mentioned authors

[20]For a thorough historical account of Zermelo's notion of *definiteness*, see Ebbinghaus (2003).
[21]For an historical account of the relationship between logic and axiomatic set theory, see Moore (1980).

thought their version was similar to, although not identical with, Zermelo's (1929) new formulation of the concept of definiteness (*Loc.cit*., fn. 2).

The current situation concerning the presentation of the axiom of separation in introductory textbooks may be divided into two broad groups: those who simply leave the concept of (well-defined) property unanalyzed, and those along the lines proposed by the Norwegian mathematician Thoralf Skolem in 1922 (Skolem 1922). After introducing the symbolism for three logical connectives or operations of negation, conjunction and disjunction together with the two quantifiers (existential $\exists$ and universal $\forall$), Skolem makes the following definition:

> By a definite proposition we now mean a finite *expression* constructed from elementary propositions of the form $a \in b$ or $a = b$ by means of the five operations mentioned (*Op.cit*., pp. 292–293, italics mine).

Notice here the word 'expression' he uses. The focus is no longer on properties but on the (logical) language used to refer to them. What is usually done in modern textbooks of set theory, is to first introduce a first-order language and then define what are its (well-formed) *formulas*, which are always finite sequences of *symbols* of the language. In the case of set theory, this language includes the primitive or basic (binary) relations of equality and membership. A variable within a well-formed formula may be free or bounded by a quantifier. For example, in the *formula* $\forall A \forall\, x \in A\ (y \in x)$ the variables A and x are bounded whereas the variable y is not, that is, the variable y is free. In this way, Zermelo´s axiom of separation became an axiom *schema* with infinitely many instances, one for each formula of the first-order language with equality and membership. It is customary to indicate, for a given formula, what its free variables are by writing them between parentheses. So, for the above example, we might denote the formula by $\varphi(y)$. In this way, the modern version of Zermelo's axiom of separation is usually stated as follows: For any given set M and any given first-order formula $\varphi(x)$, the collection $\{x \in M \mid \varphi(x)\}$ is a set. Although this version has many mathematical advantages[22], it is clearly not in the semantical spirit of Zermelo's original formulation, for he was concerned with truth values, and not (so much) with the syntax of the underlying logical language or the so-called object language. Indeed, in his 1929 paper he explicitly says that,

> What I intended to do was to derive the main theorems of set theory from the smallest possible number of assumptions and by means of the most restricted expedients. I recognized that in order to do so, the unrestricted use of "propositional functions" would be […], on account of certain "antinomies" […] dangerous. At the time, a universally acknowledged "mathematical logic" on which I could have relied did not exist—nor does it exist today when every foundational researcher has his own logistic. With my primary tasks being different, however, it hardly would have been appropriate for me to develop in extenso such a logistical foundation, particularly at a time when most mathematicians still harbored suspicions about any kind of logistic. But I believed that my explanation of the concept in question, and in particular its

[22]For one thing, it no longer makes any references to other axioms.

applications, had at least made sufficiently clear how it was meant (Zermelo 1929, p. 359).

Certainly, from the applications *Zermelo 1908a* gives of the axiom of separation, it is clear at least that "definiteness" is closed under negation, conjunction and disjunction. If $F(x)$ and $G(x)$ are any two *definite* propositional functions, then to say that the following three are also definite propositional functions

$$F(x) \wedge G(x)$$
$$\neg F(x)$$
$$F(x) \vee G(x)$$

is to say that, just like $F(x)$ and $G(x)$, each one has one and only one of the two truth values for each value of x: each one is either true or false and never both (for each value of x). And to say the latter is to say that these three connectives are just being classically defined, as it is usually done by means of the so-called truth tables.

What I propose here is to show that what the categorical version of the axiom retains from Zermelo's *semantic* formulation of the axiom in terms of definite properties, is the idea that parts of a given set M may be separated from it by means of properties, where these properties are such that the elements of *M always* either have them or don't have them (and never both!). And that as a consequence, any one of these special properties divides M into two disjunct parts which together exhaust all of M[23].

Interlude: Functions

Just as Zermelo set theory has primitive notions, so does the category-theoretic approach to the concept of set. And just as sets are almost everywhere in mathematics, *functions* too are ubiquitous. The intuitive idea behind the notion of function involves three ingredients: two collections of "things" and a correlation between the things in one collection and those in the other one. For example, let us suppose we have a group of five students and we want to write down their final grades. The group of students is called the *domain* of the function which we might call "final grade"; we usually take as the *codomain* of this kind of functions the set of numbers from 0 to 10. The codomain is thus where the function takes its values for each element of the domain. The function "final grade" then associates to each student one *and only one* number between 0 and 10. The important point here is that *each* student in the domain gets assigned *one but only one* final grade. Notice that it may happen in one specific scenario that no student gets, for example, the number 7 as his or her final grade; in such cases we still have a function from the set of students to the set {0, 1, 2, …, 10}. If instead of the set of numbers between 0 and 10, we take a larger set of numbers, say the set of *all* natural numbers, we

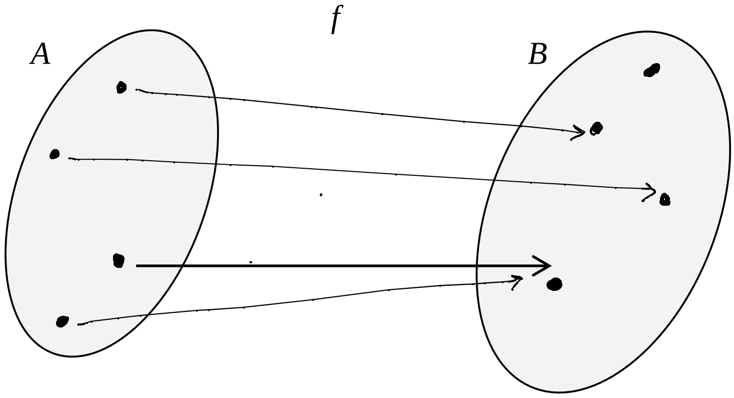
[23] See Figure 3 in the section on the categorical approach to sets.

can also define a function assigning final grades to students but this function would have a different codomain than the first one, and so it will be a *different* function, even though the "rule" for the assignment and the domain remain the same as in the first case. An analogous thing happens if we change the domain of the function; in such a case, one gets two different functions even if the codomain and the rule of assignment remain unchanged. So, one requirement for two given functions to be equal, is that they at least have the same domain and the same codomain. We will shortly see why this is so, but I will first give one example of a correlation that is *not* a function.

Let us consider for the domain *D* of our counterexample a family consisting of five members: the parents, two boys and one girl. For the codomain *C*, let us take the same family together with the immediate families of each one of the parents. If we assign to each person in *D* his or her brother, the girl in the domain will be assigned two people in the codomain *C*. Thus the rule "the brother of *x*" is not, for this particular domain, a function: at least one element of the domain is assigned to two elements in the codomain. It could also happen that for this particular domain at least one of the parents does not have a brother, and this would be another reason for the assignment "the brother of *x*" not to be a function for these particular groups of people *C* and *D*. In sharp contrast, the assignment "the father of *x*" is indeed a function from *C* to *D*.

Using Venn diagrams, we may picture a function *f* from a set *A* to a set *B* as in Figure 1.

Figure 1. *Venn Diagram of a Function from a Set A to a Set B*



And an assignment or correlation that is *not* a function from *A* to *B* could look like this

Figure 2. *Venn Diagram of a Correlation from a Set A to a Set B that is not a Function*

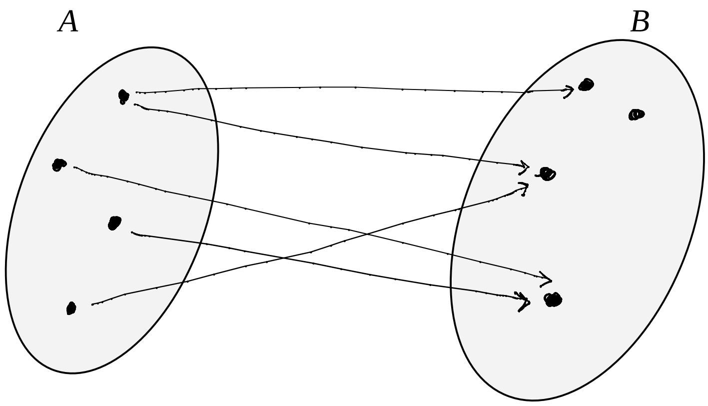


Figure 2 does not depict a function because there is (at least) one element in *A* that was assigned to more than one element in *B*, but *not* because there is an element in *B* that is not correlated with some element in *A*. The concept of function does not require that all elements of the codomain are correlated with some element of the domain.

Another way for denoting functions together with their corresponding domains and codomains is the arrow notation. So if *f* denotes a function with domain *A* and codomain *B*, we may write

$$f\colon A \to B \quad \text{or} \quad A \xrightarrow{f} B$$

If *x* is any given element of *A*, *the* member of *B* that the function *f* assigns to *x* is denoted by *f*(*x*). Usually, when the sets involved are sets of numbers, the functions mathematicians are interested in are given in terms of arithmetical operations. So, for example, if the set *A* is the set of natural numbers including the number 0, and *B* is the set of natural numbers without the number 0, we can define the function with domain *A* and codomain *B* by simply writing

$$f(x) = x + 1.$$

This function has the property that for *each element y* in the codomain *B* = {1, 2, 3, …}, there is one element *x* in the domain *A* = {0, 1, 2, 3, …} such that *f*(*x*) = *y*:

$$1 = f(0),\ 2 = f(1),\ 3 = f(2),\ \ldots,\ y = f(y - 1),\ \ldots$$

If we now change the codomain, and take instead of *B* the set *A* but keep the same domain and the same rule of assignment, the above property *f* no longer holds, namely, that each element of the codomain {1, 2, 3, …} of *f* "comes from" (*via f* itself) one element of the domain *A* = {0, 1, 2, 3, …}. For in this new case the number 0 is an element of the codomain and there is no element *x* in *A* = {0, 1, 2, 3, …} such that *x* + 1 = 0. The property that *f* has is called *surjectivity* and we say of f that it is *surjective*. Thus when we changed the codomain but retained the same domain and rule of assignment, we obtained a different function, one that, unlike f, is not surjective. Figure 1 also illustrates a function that is not surjective. This distinction gets lost in the set theoretic definition of a function as we will now see.

Functions in set theory are encoded or defined in terms of *ordered pairs*. With the notion of an ordered pair, denoted by (a, b), one can capture the idea that the element a is assigned only to the element b. If a and b are sets, then by the axiom of elementary sets, $\{\{a\}, \{a, b\}\}$ is also set. The ordered pair (a, b) is defined precisely as the set $\{\{a\}, \{a, b\}\}$. It is important to notice that this latter set is different from the set $\{a, b\}$. The idea that "a comes first" or that the order matters, is expressed in the following theorem which is easy (but somewhat tedious) to prove with the axiom of extensionality

Theorem
For any given sets a, b, c and d, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

$\square$.

We can now capture the idea that each element x of the domain of a function gets assigned to one *and only one* element of the codomain in the following terms: if $(x, b) = (x, d)$, then $b = d$. By means of an axiom called the Union Axiom[24] and the Power Set Axiom[25], it can be proved that, if A and B are sets, then the collection $\{(x, y) \mid x \in A \text{ and } y \in B\}$ is a set. Thus a function, say F, is defined in set theory as a certain set of ordered pairs, for instance $F \subseteq \{(x, y) \mid x \in A \text{ and } y \in B\}$, such that for any $x \in A$, if both (x, b) and (x, d) are elements of F, then $b = d$.

We saw earlier that the function f: $\{0, 1, 2, \ldots\} \to \{1, 2, 3, \ldots\}$ given by $f(x) = x + 1$, is surjective, whereas the function g: $\{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ given by the same rule, that is, by $g(x) = x + 1$, is not. However, according to the set-theoretic definition of function, both functions f and g are encoded in one and the same set, namely, the set

$$\{(0,1), (1,2,), (2, 3), \ldots, (x, x + 1), \ldots\}$$

So the set-theoretic account of the concept of function misses an important ingredient of it. As we will see, this is a point that the categorical approach emphasizes: every function comes equipped with both a specific domain and a specific codomain. We saw earlier that the empty set is a subset of *any* set. So, if A and B are given sets, $\varnothing \subseteq \{(x, y) \mid x \in A \text{ and } y \in B\}$, and the empty set vacuously satisfies the condition for being a function from A to B for *any* sets A and B. This is an extreme example that shows how the definition of the concept of function as a certain subset of ordered pairs neglects the importance of taking into account what

[24]This axiom states that for any given set M the collection of all the elements of the elements of M is also a set.
[25]This postulate guarantees the existence, for any set M, of the set of all of the subsets of M.

the domain and the codomain are: the empty set is a function from *any* set to *any* set.

There is another significant property a function may have. Consider the following two sets:

$$A = \{1, 2\} \text{ and } B = \{3, 4\}$$

and let $f: A \to B$ and $g: A \to B$ be defined as follows: $f(1) = f(2) = 3$, $g(1) = 3$ and $g(2) = 4$. One difference between these two functions is that $f(1) = f(2)$ even though $1 \neq 2$, whereas $g(1) \neq g(2)$. In other words, for all elements x and y in the domain A, if $x \neq y$ then $g(x) \neq g(y)$. This property is called *injectivity*. So the function f is *not* injective. A function that is both injective and surjective is called *bijective*. Thus the function g is bijective. Using the same sets A and B from this example, we can define another bijective function: let $h: A \to B$ be given by $h(1) = 4$ and $h(2) = 3$. Figure 1 is an example of a function that is not injective.

Bijective functions are quite useful in many ways. For instance, they allow us to determine, without counting, whether any two given sets have the same number of elements. A striking example is given by the set N of natural numbers $\{0,1, 2, \ldots\}$ and the set E of even numbers $\{0, 2. 4. \ldots\}$. Let $f: E \to N$ be given by $f(x) = x \div 2$. This is indeed an injective function (recall that all basic arithmetical operations are functions). Let, then, $y \in N$. In order to show that f is surjective we need to find an element x of E such that $f(x) = y$. Well, since $y \in N$, then $2y$ is an even number, and so it is an element of E. Moreover, $f(2y) = 2y \div 2 = y$. And this shows that f is also surjective. So, despite appearances to the contrary, there are as many natural numbers as there are even numbers.

But bijective functions are also important from an epistemological point of view, a fact that we will use extensively when presenting the categorical version of the axiom of separation. It often happens, like in the above example, that the bijective function is given or defined in such a way that it allows to *construct* an object or element of the codomain *from* an element of the domain, *and vice versa*, even when these elements are not numbers but more complicated mathematical objects. And this in turn allows to rethink or reconceptualize a mathematical object, sometimes in simpler ways that are more amenable to mathematical calculations. In this way, one can move back and forth from two different ways of conceiving of some mathematical object, according to one's needs.

Let us consider now an arbitrary set A. We can always define a function with domain and codomain the set A itself, namely, the function $f: A \to A$ given by $f(x) = x$ for all $x \in A$[26]. This function is called *identity on A* and it is usually denoted by "id_A". And given any two functions where the codomain of one is the same as the domain of the other function

$$f: A \to B \quad \text{and} \quad g: B \to C$$

[26]If the set A is empty, then this is vacuously true.

we can always define another function, called their *composition*, with domain A and codomain C. For suppose that $x \in A$. Therefore $f(x) \in B$ and hence we can apply g to this element of B, $g(f(x))$, obtaining in this way an element of C:

$$gf : A \rightarrow C$$
$$fg(x) = g(f(x)) \in C$$

For example, let us consider again the set of natural numbers $N = \{0, 1, 2, 3, \ldots\}$ and the set E of even numbers $\{0, 2, 4, 6, \ldots\}$. Let f: $N \rightarrow E$ and g: $E \rightarrow N$ be defined as follows for each $x \in N$ and each $y \in E$

$$f(x) = 2x \quad \text{and} \quad g(y) = y + 3$$

Let us now take, say, the element $7 \in N$. Then

$$f(7) = 14 \text{ and hence } g(f(7)) = g(14) = 14 +3 = 17.$$

More generally, for any $x \in N$, $fg(x) = 2x + 3$. And this is certainly a function from N to N since there is no natural number x for which $2x + 3$ would give us two *different* results.

Let us again consider two arbitrary functions f: $A \rightarrow B$ and g: $B \rightarrow C$ and suppose we also have a third one, say h: $C \rightarrow D$, where D is just another arbitrary set. Since the codomain of the composition $gf : A \rightarrow C$ is the same as the domain of h, we can again compose these two functions

$$h(gf): A \rightarrow D.$$

But we can also consider the composition of just g: $B \rightarrow C$ g: $B \rightarrow C$ with h: $C \rightarrow D$:

$$hg: B \rightarrow D.$$

And we can now compose f: $A \rightarrow B$ with hg: $B \rightarrow D$, thus obtaining another function from A to D:

$$(hg)f : A \rightarrow D.$$

When the sets involved are sets of numbers and the functions are defined by the usual basic arithmetical operations, $h(gf)$: $A \rightarrow D$ and $(hg)f : A \rightarrow D$ are the same function. However, when one takes the concept of *function* as primitive (as it is done in the categorical approach), the equality $h(gf) = (hg)f$ must be postulated as an axiom. The equality $h(gf) = (hg)f$ is called *associativity of composition*.

Finally, let A, B and C be given sets, and let id_B: $B \rightarrow B$, f: $A \rightarrow B$ and g: $B \rightarrow C$. We then have the compositions

$$(id_B)f : A \to B \text{ and } g(id_B): B \to C.$$

Consider now an arbitrary $x \in A$ and an arbitrary $y \in B$. Then, by the definition of identity functions, we have the following equalities

$$(id_B)f(x) = f(x) \text{ and } g(id_B(y)) = g(y).$$

That the equalities $(id_B)f = f$ and $g(id_B) = g$ actually hold, is also postulated as an axiom in the categorical approach. Existence of composition (as long as the codomain of one function coincides with the domain of the other one), existence of identities for every set, associativity of composition, and these two last equalities constitute the basic axioms when the concept of function is taken as primitive. As we shall see in the next section, there are other axioms that characterize the category of sets (and thus distinguish it from other categories). But before presenting the categorical approach, we need to look at a few more special kinds of functions.

Let us consider an arbitrary set A, a set with just one element, for example, the set $\{\varnothing\}$, and a set with exactly two elements, for instance, the set $\{\varnothing, \{\varnothing\}\}$. Let us call '$t$' any one of the elements of the set $\{\varnothing, \{\varnothing\}\}$. For example, we can le $t = \varnothing$. We can now define a function f with domain A and codomain the one-element set $\{t\} = \{\varnothing\}$ by simply letting $f(x) = t$ for all $x \in A$. Moreover, this is the *only* function from A to the set $\{t\}$ since there are no other elements in $\{t\}$ that we can assign to the elements of A. This result is in fact more general: for *any* one-element set and any given set A, there is always one *and only one* function from A to the one-element set.

Let us now consider functions from the one-element set $\{t\}$ to A. We have that, for each $x \in A$, we can define a function f_x: $\{t\} \to A$ by letting $f_x(t) = x$. But conversely too. Each function g: $\{t\} \to A$ determines a unique element of A, namely, $g(t) \in A$. This *bijective* correspondence between functions from a one-element set to A and elements of A, is the point of departure for reconceptualizing, in categorical terms, the notion of element, as we will see in the following section.

Suppose now that we are given a function f from A to the two-element set $\{t, \{\varnothing\}\}$. By Axiom III, the collection of all $x \in A$ such that $f(x) = t$ is a subset of A, even when there is no $x \in A$ such that $f(x) = t$, for in this case the subset of A in question would be the empty set. At the other extreme, if $f(x) = t$ *for all* $x \in A$, then the subset in question would be A itself. Intuitively, all the cases in between these two extremes, should give us *all* of the subsets of A. In other words, there is a close connection between *all* functions from A to the two-element set $\{t, \{\varnothing\}\}$ and subsets of A. Indeed, given a subset S of A, we can always define a function f_S from A to the set $\{t, \{\varnothing\}\}$: for any given $x \in A$, we simply let $f_S(x) = t$ if and only if $x \in S$. This (bijective!) correspondence between subsets of A and functions from A to a two-element set is at the heart of the categorical version of the axiom of separation, as we will shortly see.

There is one more relationship we need to look at between the concept of function and the notion of subset. Consider any two sets A and B. If A is a subset of B, we can always define a function $i: A \rightarrow B$ by simply letting $i(x) = x$ for all $x \in A$. And this function is clearly injective. But moreover, if we are given an injective function, say $j: C \rightarrow B$, then it determines (by Axiom III) a subset of B, namely the set

$$\{y \in B \mid \text{there is } x \in C \text{ such that } y = j(x)\} = \{\, j(x) \in B \mid x \in C \,\}.$$

Thus we have a correspondence between, on the one hand, subsets of a given set and, on the other hand, injective functions with codomain the set in question[27]. And this correspondence is bijective. It will be useful to keep in mind this observation when we come, in the next section, to the definition of *subset* in the categorical approach.

The Category-theoretic Approach to Sets

Category theory is a branch of mathematics that is nowadays at the forefront of developments in the foundations of mathematics[28]. It is a highly abstract theory so I hope this section can also serve the purpose of giving a first approximation to it.

As mentioned earlier, the categorical approach starts from axiomatizing the concept of *function* or, as I will call it from now on, *map*. Maps always come "attached" to two objects[29], one called its *domain* and the other one its *codomain*. So the concepts of domain and codomain are also primitive. Certainly, any given object can be the domain of one map but the codomain of another map. The point is that the properties of these domains and codomains will be given, not in terms of their elements, but in terms of maps from and to them[30]. As it is usual in category theory, either notation:

$$f\colon A \rightarrow B \quad \text{or} \quad A \xrightarrow{f} B$$

stands for "f is a map with domain A and codomain B".

The categorical approach to set theory starts then from what in *LR* are called *abstract sets* (or simply *sets*) and certain maps between them. The first three axioms capture the basic properties of functions presented in the previous section.

[27]Notice that the set $\{\, j(x) \in B \mid x \in C \,\}$ is a subset of B, not of C, where B is the codomain of the injective function $j: C \rightarrow B$.

[28]The title of Landry (2017) is a good testimony to this. It also contains a wealth of references concerning relatively recent debates on categorical foundations for mathematics.

[29]The word 'object' in this context is a technical term referring exclusively to the domains and codomains of maps or, as they are called in general, 'morphisms' of a category.

[30]In fact, not all categories are categories *of sets*. For example, given *any* category, one can always form the category whose objects are the maps or, more generally, morphisms of the given category.

Axiom (Existence of Composition and Identity Maps)

Every map has associated a set called its domain and a set (not necessarily distinct from the domain) called its codomain.

For any given maps $f: A \to B$ and $g: B \to C$, there is a map $gf: A \to C$ called their *composition* (there may of course be other maps from A to C besides the composition map gf).

For any given set A, there is a map $id_A : A \to A$, called the *identity map on A* (besides this identity map, there may also be other maps from A to A).

Axiom (Associativity of Composition)

For any given maps $f: A \to B$, $g: B \to C$ and $h: C \to D$, $(hg)f = h(gf)$.

Axiom (Identities are Neutral with Respect to Composition)

For any given map $f: A \to B$, $f(id_A) = f$ and $(id_B)f = f$.

Axiom (Terminal Set)

There is a set, denoted by 1, with the property that for any given set A there is one *and only one* map $A \to 1$ from A to 1. This special map is denoted by the symbol $!_A$.

The set-theoretic counterparts of this set 1 are, as we saw in the previous section, the one-element sets. In that section we also saw that there is a bijective correspondence between functions from one-element sets to an arbitrary set A and elements of A. So we have the following

Definition (Elements of Sets)

Let A be an arbitrary set. Then an *element* of A is a map $1 \to A$[31].

By the definition of the set 1, there is only one map from 1 to itself, the special map denoted by $!_1$; $1 \to 1$. But by an axiom, there is also the identity map id_1: $1 \to 1$ from 1 to itself. So, the special map $!_1$ and the identity map id_1 must be the same. In this way we arrive at our first theorem:

Theorem

The set 1 has exactly one element.

□.

[31]One interesting consequence of this definition, which I hope can serve as a motivation for the concept defined below of 'membership in a part of a set', is that if A and B are different, then they cannot have any elements in common: for if a: $1 \to A$ is an element of A, it cannot be equal to any element $1 \to B$ because the codomains of these maps are, by hypothesis, different from each other—recall that a necessary condition for two functions to be equal is that they at least have the same domain and the same codomain.

For the counterpart of *subset* in the category-theoretic approach, we need the following definitions:

Definition (Monomappings)

Let A and S be given sets and let $m: S \to A$ be a given map. Then m is a *monomapping* if the following holds: for any given set T and any given maps $f_1: T \to S$ and $f_2: T \to S$

$$\text{if } mf_1 = mf_2 \text{ then } f_1 = f_2.$$

If in the above definition we take T as the set 1, we arrive at the categorical version of *injective function*. The generality of the definition is due to the fact that concepts in category such as that of *monomapping* are defined for arbitrary categories, so in particular for categories that may not have a terminal set or, even more generally, that are not about sets at all.

In the previous section we saw that for any given set B there is a bijection between its subsets and injective functions with codomain B. We now make use of this bijection to reconceptualize in terms of maps the notion of subset.

Definition (Parts of Sets)

Let A be a given set. Then a *part* of A is a monomapping with codomain A.

Since identity maps are monomappings, every set is a part of itself.

For any given set A, maps in general with codomain A are called *generalized elements* of A and they contrast with what are called *global elements* of A, that is, maps with domain a terminal object and codomain A. The definition of *membership in a part* (like the definition of monomapping), in *LR* is given for generalized elements, partly because it is meant to apply in arbitrary categories. However, and for the purposes of keeping close the analogy with Zermelo's axioms, I will confine the following definition to the case of maps with domain the terminal object 1.

Definition (Global Membership in a Part)

Let A and U be given sets, $a: 1 \to A$ an element of A and $m: U \to A$ a part of A. Then we say that a is a *member* of the part m of A, if there is an element of U

$$\boldsymbol{m_a}: 1 \to U$$

such that the composition $m\boldsymbol{m_a}: 1 \to A$ is equal to the map $a: 1 \to A$.

When an element $a: 1 \to A$ is a member of a part $m: U \to A$ of A, we write $a \in_A m$. We could think of the above definition as saying that if the element $a: 1 \to A$ of A is a member of the part $m: U \to A$ of A, then there is an element $\boldsymbol{m_e}: 1 \to U$

of U that m interprets as the element a: $1 \to A$ of A in the sense expressed by the equality $m\boldsymbol{m_a} = a$. The map $\boldsymbol{m_a}$: $1 \to U$ may also be thought of as a "witness" of the fact that a: $1 \to A$ is indeed a member of m: $U \to A$.

In order to be able to incorporate Zermelo's semantic notion of "definite property" *within* the theory, that is to say, without appealing to syntactic notions such as "expressions in a first-order language", we need the following

Axiom (Truth-value Set)

There is a set, denoted by 2, with exactly two elements, t: $1 \to 2$ and b: $1 \to 2$[32].

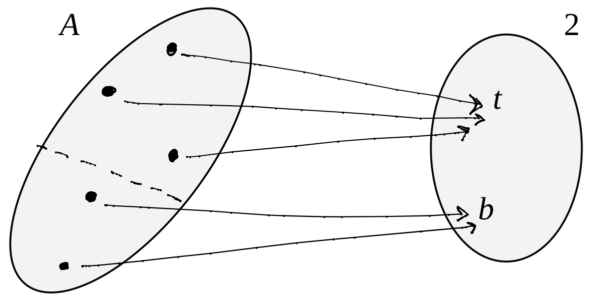
The key idea is to now think of the elements of 2 as truth-values; that is why one of them (it really does not matter which one) is usually called t for "true". Now, in order to capture the bivalence of Zermelo's definite properties, we make the following

Definition (Properties of Elements of a Set)

Let A be a given set. Then a *property* of elements of A is a map from A to 2.

Notice that what this definition implies is what Zermelo's definite properties *do*: they divide a set into two mutually disjoint parts which together exhaust the whole set; one part consists of those elements of the set that are assigned the map or value t, that is, those that do have the property in question, and those elements that are assigned the other element or value b: $1 \to 2$ of 2, that is, those elements that do not have the property in question, as the following diagram illustrates

Figure 3. *Venn Diagram of a Property of Elements of a Set A*



Given a set A, an element x: $1 \to A$ and a property φ: $A \to 2$ of elements of A, to say now that x has property φ is to say that the composition φx: $1 \to 2$ is equal to the element "true" t: $1 \to 2$ of 2. By simply being maps with codomain this special object with exactly two elements, it is always definite *tout court* whether any given element of the domain has or does not have the property in question. More precisely, the composition

[32]The formulation in *LR* of this axiom is different. I changed it since the original one requires more concepts than are necessary for the exposition. See *LR*, pp. 19, 27–28.

$$\varphi x\colon 1 \to 2$$

has the same domain and the same codomain as the only two elements of the set 2. So the map $\varphi a\colon 1 \to 2$ *must be* equal to one and only one of these two elements of the set 2. In other words, it must be equal either to the truth value *true* or to the truth value *false*. In the first case, we will say that the element $a\colon 1 \to A$ of A has the property $\varphi\colon A \to 2$, and in the other case we will say that it does not.

As we saw in the previous section, there is a bijection between subsets of any given set M and functions from M to any two-element set, functions which now in this context correspond to properties of elements of M. We will now characterize this bijection in more precise terms. Consider then a part $i\colon U \to A$ of A and a property $\varphi\colon A \to 2$ of elements of A. What we want to capture *solely* in terms of maps is the idea that $a\colon 1 \to \mathrm{A}$ is a member of the part $i\colon U \to A$ if and only if $a\colon 1 \to A$ has the property $\varphi\colon A \to 2$. And to say in terms of maps that $a\colon 1 \to A$ has the property $\varphi\colon A \to 2$ is to say that the composition $\varphi a\colon 1 \to 2$ is equal to the truth value true $t\colon 1 \to 2$. In other words, we simply say that the equality $\varphi a = t$ holds, which in turn we can rephrase by saying that the statement "a has property φ" is true. The following definition is given in *LR* in terms of generalized elements but, again only for the purposes of this discussion, I give it only terms of global elements. The definition is the category-theoretic way of expressing that "the element a of A has a certain property φ if and only if a is a member of the part $i\colon U \to A$".

Definition

Let A be a given set, $i\colon U \to A$ a part of A and φ a map from A to the set 2. We say that $\varphi\colon A \to 2$ is a *characteristic function* of the part $i\colon U \to A$ if for any element $a\colon 1 \to A$ of A, the following holds

$$a \in_A i \text{ if and only if } \varphi a = t$$

In other words, a map or property $\varphi : A \to 2$ is a characteristic function of a part $i\colon U \to A$ of A if, for any element $a\colon 1 \to A$ of A, $a \in_A i$ if and only if a has the property φ. We can now state the category-theoretic version of Zermelo's axiom of separation.

Axiom (Properties and Parts of Sets)

Let A be a given set. Then any property $\varphi : A \to 2$ of elements of A is the characteristic map of a part of A. And any part of A $i\colon U \to A$ has a unique characteristic map $\varphi_i : A \to 2$[33].

[33]Given a property $\varphi\colon A \to 2$, the part of A corresponding to it is sometimes denoted as $\{x \mid \varphi\}$ (see, e.g., Goldblatt 1984, p. 107).

This axiom expresses *exclusively* in terms of the semantic notion of map, the bijection that, as we saw, exists (via the axiom of separation) within Zermelo's set theory between subsets of a given set and functions from it to a two-element set. It thus establishes a clear and close connection between properties of elements of a set with its subsets or parts, the connection I believe Zermelo intended to express with his notion of *definite property*.

The axiom that characterizes the particular set 2 is a special case of a more general concept called *subobject classifier* which some categories have and some do not. I am not claiming that this concept arose with the purpose of clarifying Zermelo's notion of definite property, it did not (see e.g., McLarty 1990). What I am claiming is that by taking seriously Zermelo's emphasis on the semantic aspect of the axiom of separation, one can see the particular subobject classifier:

$$1 \xrightarrow{t} 2 \xleftarrow{b} 1$$

as removing the imprecision in Zermelo's notion of definite property while at the same time keeping its semantic spirit, thus showing an instance of mathematics' own process of evolution. For category theory arose almost 40 years later than Zermelo's axiomatization and it was not until 1964 when the first categorical approach to set theory was published. Moreover, the last axiom does not, unlike Zermelo's axiom of separation, make any references to a background logic nor to other axioms of the theory (except, of course, an implicit but necessary reference to the axioms characterizing the sets 1 and 2).

Concluding Remarks

As mentioned previously, it was not until 1929 that Zermelo responded to the proposed solutions concerning the vagueness in his concept of definiteness. Regarding a solution like Skolem's, Zermelo writes:

> […] It is not concerned with the proposition *itself* but its *formation* or *generation* […] It *runs counter* to the nature of the axiomatic method and therefore it is really, in my opinion, just as out of place in any axiomatics […] (Zermelo 1929, p. 363, emphasis in the original).

What Zermelo wanted was an axiomatic characterization of the concept of definiteness, where 'axiomatic' contrasts with what he called 'genetic' or 'constructive'. Genetic accounts of the notion of definiteness are precisely those that are concerned with the formation of *expressions* of an underlying object language. So what he did was to give axioms characterizing the concept of definiteness as a system closed under the logical connectives and the quantifiers. Using modern notation, this system is axiomatized as follows[34]:

[34] I am omitting the clause concerning quantification over propositional functions since it seems unnecessary for the purposes of this discussion.

Let B be a domain and R a system of *fundamental relations* written as $r(x, y, z, \ldots)$, where the variables $x, y, z, \ldots$ range over the domain B. Then we say of a proposition p that it is "*definite with respect to R*", and write Dp, if the following is the case:

I. $Dr(x, y, z, \ldots)$ for every relation r of R and any variables $x, y, z, \ldots$ ranging over the domain B. In other words, all fundamental relations are definite.
II. If Dp holds, then also $D(\neg p)$ holds. Definiteness is closed under negation.
III. If both Dp and Dq hold, then so do $D(p \wedge q)$ and $D(p \vee q)$. Definiteness is closed under conjunction and disjunction.
IV. Let $f(x, y, z, \ldots)$ be a propositional function with free variables $x, y, z, \ldots$ ranging over the domain B. If $Df(x, y, z, \ldots)$ holds for all $x, y, z, \ldots$, then so does $D(\forall x, y, z, \ldots f(x, y, z, \ldots))$. Likewise, if $Df(x, y, z, \ldots)$ holds for all $x, y, z, \ldots$, then so does $D(\exists x, y, z, \ldots f(x, y, z, \ldots))$. Definiteness is closed for propositional functions under universal and existential quantification.
V. If P is the system of all "*definite*" *propositions*, then it has no proper subsystem P_1 that, on the one hand, contains all fundamental relations from R, while already including, on the other hand, all negations, conjunctions, disjunctions and quantifications of its own propositions and propositional functions. Thus P is the largest system containing all and only definite propositions. Any proposition not obtained by this inductive procedure would not count as definite[35].

It is clear that this axiomatization of a system of definite proposition is the semantical counterpart of the inductive characterization of a first-order formula or expression, for we should bear in mind that Zermelo's "pre axiomatic" notion of definite proposition is still semantic, that is, even 21 years after the publication of his "Investigations in the foundations of set theory I", a definite proposition is for him one that is either true or false and never both (and of course, what determines what its truth value is, are the axioms and the universally valid laws of logic). His thorough reluctance to place the focus on the syntax, together with his insistence on *the* truth value a definite proposition *always* has, makes Zermelo's axiom of separation close in spirit to its categorical formulation. It is unfortunately beyond the scope of this paper to explain how, given two properties $\varphi, \psi \colon A \rightarrow 2$ of elements of A, their corresponding negations, their conjunction and their disjunction are also properties of elements of A and hence correspond to certain parts of A[36].

[35]Before giving this axiomatization, Zermelo briefly discusses models of axiomatic systems and says that definiteness is "what is decided in every single model" (Zermelo 1929, p. 361). Although models play no role in his axiomatic characterization of the notion of definiteness, this passage supports my claim that this notion remained for him thoroughly semantic.
[36]Needless to say, the case of the quantifiers is even more complicated, but I believe it can be specialized to the category of sets and presented along the lines proposed here.

I suggest then that by looking at other of Zermelo's axioms from 1908 with modern mathematical tools such as category theory, one may glimpse a continuum of ideas rather than a straightforward repudiation of some of them in favor of new ones. Moreover, it seems to me that this way of looking at old mathematics may also serve the purpose of teaching and learning new mathematical concepts and theories. The case of category theory is of particular importance within philosophy for it helps us to take a general and novel look at various parts of mathematics, including set theory.

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