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# Epicurean Induction and Atomism in Mathematics

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*In this paper[1], we explore some positive elements from the Epicurean position on mathematics. Is induction important in mathematical practice or useful in proof? Does atomism appear in mathematics and in what ways?*

**Keywords:** *Epicurus, induction, Polya, proof, atomism*

## Introduction

The Epicureans, in general, considered geometry and mathematics only for utility and practical purposes. They regarded abstract mathematics useless and they did not, overall, expect or encourage their members to do any mathematics beyond perhaps some very basic level.[2] (White 1989, pp. 297–298). There were, of course, some Epicureans quite knowledgeable in mathematics, such as Polyaenus, Philonides, Zeno and Demetrius. Also, the Epicureans did not have mathematics or logic among their primary philosophical interests or teachings.[3] (White 1989, pp. 297–298, Cicero 1914, p. 25). Their belief that all knowledge is empirical and the inductive logic that guided their philosophy, do not seem to align with some of the most important aspects of mathematics, such as abstraction, deduction and proof.

Furthermore, Epicureans are well known for their atomism in physics. Interestingly, some scholars ascribe to Epicureans a type of "mathematical atomism"[4] (Sedley 1976, Mau 1973, White 1989) that suggests indivisible theoretical minima in the atom which serve as units of measurement of the atom (Sedley 1976, p. 23). Epicurus speaks of such minima in the 'Letter to Herodotus' (Inwood and Gerson 1988, p. 10). Even though is not completely clear what he means by 'minimal parts in the atom', it seems only reasonable to ascribe mathematical atomism to the Epicureans. That is because they explicitly rejected infinite divisibility and their conceptual atomism is insinuated in their critique of Euclidean geometry by expressing skepticism as to whether two lines would intersect at a point and not a small segment instead (Aristidou 2017, 2020a). Nevertheless, the sources do not show that the Epicureans formally developed any atomistic geometry.

In the following sections, we argue that even though the Epicureans aimed at discrediting mathematics in many ways, they can nonetheless be found accrediting mathematics indirectly in two ways: (a) Inductive inference helps in deductive mathematical proofs by establishing hypotheses and conjectures. (b) mathematical atomism appears in some ways in mathematics and it is important and useful. It is important to emphasize that the Epicureans did not hold the view that induction is

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an important feature of proof or the view that theoretical atomism appears sometimes in mathematics and it is important, but rather that the Epicureans had the relevant conceptual resources (e.g., inductive inference, minima, etc.) to provide a foundation to some aspects of mathematical practice, such as proof, atomistic geometry, etc., had they been interested in doing so.

## Epicurean Logic

Behind Zeno's methodological critique of the foundations and logical structure of geometry was the background of Epicurean logic, a quite distinct logic in ancient Greece. What constituted Epicurean logic, its range, uses, etc., had a direct effect on their stance on mathematics. And, ultimately, that distinct Epicurean logical point of view seems to have been beneficial for mathematics and proofs in particular.

Logic was not part of the Epicurean Canon, namely the epistemological rule-set and conditions on how truth is evaluated. Yet, they neither rejected logic altogether nor were ignorant of it, as some of their critics claim (e.g., Seneca, Cicero) (Inwood and Gerson 1988, pp. 38, 52), but accepted aspects of logic under certain conditions and adapted it to meet their epistemological beliefs. For example, Philodemus' 'On Signs' (De Lacy and De Lacy 1941) is a work on logic. The Epicurean logic could be roughly summarized as follows:

**(A) Accepted**: *(i) sound arguments, (ii) contrapositive (by contradiction), (iii) induction, (iv) abduction.* More specifically:

(i) *sound arguments*: they accepted $p \to q$, when $p$ is real/fact. (Inwood and Gerson, 1988, p. 5); Sedley 1982, pp. 242–244). (similar to the Stoic (Chrysippus) "συνάρτησις"). (Sedley 1982, p. 245).
(ii) *contrapositive*: they accepted $\neg q \to \neg p$, to go from 'πρόδηλον' (obvious/ observed fact) to 'άδηλον' (not obvious/unobserved fact). (Inwood and Gerson 1988, p. 58, Stocks 1925, p. 195). Also, they accepted the by contradiction method when something contradicts with the facts (Inwood and Gerson 1988, p. 9).
(iii) *induction*: Epicurus, but even more so Zeno and Philodemus, developed a theory of inductive inference[5] which bases the inference on empirical observation. (Marquand 1883, pp. 1–11). An inference is justified if it is verified by the facts ("επιμαρτύρησης") or is formed by analogy to other facts ("κατ' αναλογια τρόπος"). An inference is not justified if it contradicts the facts ("αντιμαρτύρησης").
(iv) *abduction*: they accepted something if it facilitates explaining real facts (Inwood and Gerson 1988, p. 9).

**(B) Rejected**: *(i) Principle of Bivalence (PB), (ii) Law of Excluded Middle (LEM), (iii) abstractions, (iv) dialectic.* More specifically:

(i) *PB*: they rejected this principle and adopted the "multiple method" (πλεοναχός τρόπος) which suggests multiple explanations or reasons for something (Inwood and Gerson 1988, p. 37).

(ii) *LEM*: rejected this principle for future propositions and in defense of free will. (Inwood and Gerson 1988, p. 42).
(iii), (iv) *abstractions, dialectic*: detested abstractions and technicalities, deduced things from facts (Crespo 2014, p. 1, Stocks 1925, p. 188).

Hence, in a sense the Epicureans adopted and promoted a more "realistic/ pragmatic" logic which contained the seeds for later modern logics such as Mill's empirical logic (Marquand 1883, p. 1), relevance logic[6] (Sedley 1982, pp. 247–248) and Pierce's semiotic logic (Belucci 2016, pp. 261–262).

Here, we take a closer look on some matters from (A), especially (A) (iii) (i.e., induction) which was a main feature of Epicurean logic. Other matters from (B) hopefully could be discussed in a future paper.

It is not very difficult to see that the Epicurean logic is not the most appropriate logic for mathematics. Truth in mathematics has stronger conditions and it is tied to proof. So, insisting on contrapositive and induction does not guarantee proof. We give two examples:

**(1)** ***The contrapositive is not always useful in proof.*** Consider the following two theorems:

(i) "If $a$ and $b$ are rationals, then $a + b$ is rational". The proof here is direct. If one tries to prove it by contrapositive/contradiction, i.e., by supposing that $a + b$ not rational, then it does not lead anywhere.

(ii) "If $n$ is even, then $n^2$ is even". Just like the example above, the proof here is also direct. If one tries to prove it by contrapositive/contradiction, i.e., by supposing that $n^2$ is not even (i.e., it is odd), then it does not lead anywhere.

**(2)** ***Induction helps in proof but does not give a proof.*** Consider the following sequence of natural numbers (Jones 2011):

$$12, 121, 1211, 12111, \ldots$$

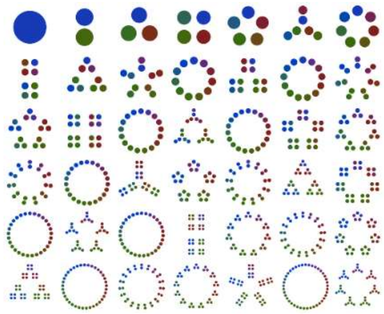
That is, the sequence formed by starting with 12 as a first term and then adding a 1 on the right of every other term. It turns out that all these numbers up to $12\underbrace{11...1}_{136}$ are composites, but the number right after is a prime![7] So, according to Epicurean logic, the statement:

"the sequence 12, 121, 1211, 12111, … consists of composite numbers"

should be true because it inductively satisfies the Epicurean requirement of επιμαρτύρησης, i.e., that it is confirmed by a large number of examples without something pointing to the contrary. Yet, the statement is false.

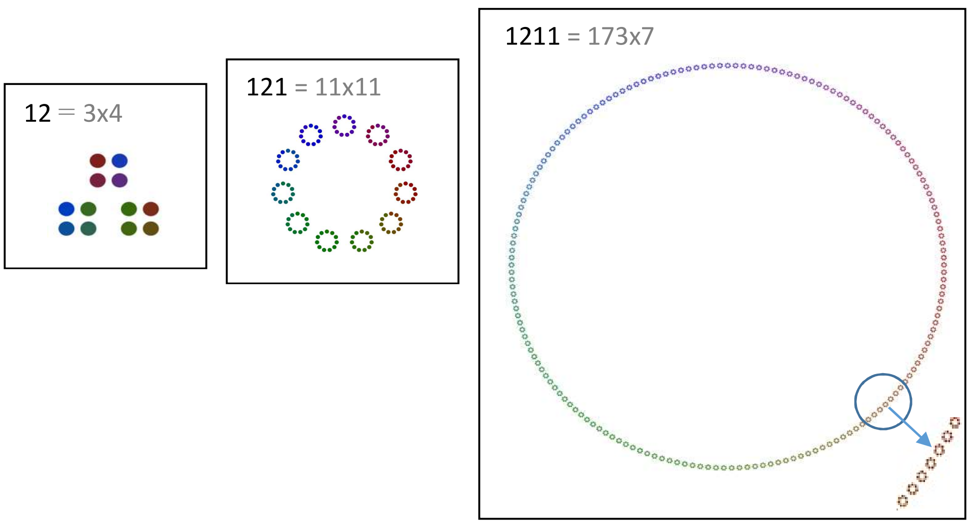
Of course, the Epicureans could have dismissed the examples above as irrelevant, and maintain that induction applies to real things (things of the senses) and not abstract objects such as the numbers 12, 121, 1211, 12111, … given above. But, one could give a visual representation of composite or prime numbers and challenge the Epicureans once again on the limitations of induction as a proof method in mathematics. For example, consider the following representation of natural numbers (Dancstep 2015):

**Figure 1.** *Primes*



In Figure 1, primes greater than 3 are represented as rings of dots. Composites have other shapes, but not rings of dots (see Figure 2).

**Figure 2.** ***Composites***




One could empirically see[8] here (Figure 2) that 12, 121, 1211, from the previous example, are composite. So, not everything in mathematics is abstract jargon. An Epicurean could also perhaps check empirically that the next 10 or 100 or 1000 cases after 1211 are also composites. Yet, our senses can only carry us that far. Because, $12\underbrace{11...1}_{136}$ certainly cannot be checked in a person's lifetime.[9]

So, the problem with the Epicurean logic is that: (a) It is not appropriate for mathematics, yet mathematics is important (socially, philosophically, educationally, etc) (b) It has to dismiss mathematics as unscientific due to its nature and methods, yet mathematics is important and necessary for science. It also, in a way, works like science. (c) It has to dismiss mathematics for empirical reasons due to its abstraction, yet mathematics can take us where our senses cannot (e.g., x-rays, microscopes, telescopes, zoom, etc.). But, could Epicurean logic be useful in mathematics?

## Induction in Mathematical Practice

As we have seen already, Epicurean empiricism and the inductive logic that guided their philosophy do not seem to align with some of the most important aspects of mathematics (such as abstraction, deduction and proof). Yet, could empiricism and induction be relevant or even useful in mathematics? They could indeed, because even though deductive inference is central in proofs and in mathematics in general, it is not the only type of inference in mathematical practice.

Peirce considers three kinds of logical inference, namely deductive, inductive and abductive, which he sees as important stages in mathematical inquiry (Bellucci and Pietarinen 2015). Certainly, deduction allows one to move from some hypotheses to a conclusion, but hypotheses and conjectures must be formed in the first place. That can be done by induction and abduction by looking at some specific examples first, draw analogies, conjecture and then generalizing. In other words, the scientific (i.e. Epicurean) aspect of proof is as important as the formal one.

Deduction, in mathematical inquiry, usually comes at the last stage as a way to verify certain observations. Polya (1973) and Lakatos (1976) explain the process of mathematical discovery very clearly. For example, Polya lays down some steps for general problem solving that include: understanding the problem, experimenting, conjecturing, generalizing and proving or disproving. The steps before the proving step are what one would call the inductive/abductive stage.

To emphasize the importance of induction in a mathematics, Polya gives a nice example applying all the previously mentioned steps (Polya 1973). In particularly, he uses the well-known theorem - "*The Sum of the First n Cubes is a Square*"- to make his point, showing all the previous steps and activity that led one to the theorem, doing calculations, using visuals, forming conjectures, etc. The relevant passages from Polya are given below, on which we underline the most characteristic points and comment on briefly.

Firstly, Polya points out that induction and mathematical induction are different processes. Nevertheless, they share some common ground and are both used in mathematics. The interesting point is that induction is used in mathematics too and, as he says later, it is also important. Then, he proceeds with the first crucial observations. Characteristically:

> **Induction and mathematical induction.** Induction is the process of discovering general laws by the observation and combination of particular instances. It is used in all sciences, even in mathematics. Mathematical induction is used in mathematics alone to prove theorems of a certain kind. It is rather unfortunate that the names are connected because there is very little logical connection between the two processes. There is, however, some practical connection; we often use both methods together. We are going to illustrate both methods by the same example.
>
> 1. We may observe, by chance, that
>
> and, recognizing the cubes and the square, we may give to the fact we observed the more interesting form:
>
> $$1^3 + 2^3 + 3^3 + 4^3 = 10^2.$$
>
> How does such a thing happen? Does it often happen that such a sum of successive cubes is a square?
>
> (Polya 1973, pp. 114–121).

More specifically, induction is the thought process in which based on specific observations and evidence one claims a general conclusion or law. Mathematical induction is a method for proving that a mathematical statement is true for all natural numbers, and it involves two steps: (a) the "base step" in which one shows that the statement is true for some initial special cases and (b) the "inductive step" in which one proves that if the statement is true for the $n^{\text{th}}$ case, then it is also true for $(n+1)^{\text{th}}$ case. The similarity of the two processes is that both begin by checking particular initial cases, and use them to generalize. But, the crucial difference is in the way the two processes "use" the particular cases to establish truth for the general case. Induction claims truth based on the number and strength of the evidence but does not establish it. Mathematical induction establishes truth deductively, i.e., it proves that given the evidence the general must follow.

After Polya's crucial observation that $1^3 + 2^3 + 3^3 + 4^3 = 10^2$, i.e., that the first four cubes add up to a square, he proceeds to check some more nearby cases just as a scientist would do in order to see if the evidence would lead to a general conjecture:

> What can we do for our question? What the naturalist would do; we can investigate other special cases. The special cases $n = 2, 3$ are still simpler, the case $n = 5$ is the next one. Let us add, for the sake of uniformity and completeness, the case $n = 1$. Arranging neatly all these cases, as a geologist would arrange his specimens of a

certain ore, we obtain the following table:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1+8 &= 9 = 3^2 \\ 1+8+27 &= 36 = 6^2 \\ 1+8+27+64 &= 100 = 10^2 \\ 1+8+27+64+125 &= 225 = 15^2. \end{aligned}$$

It is hard to believe that all these sums of consecutive cubes are squares by mere chance. In a similar case, the naturalist would have little doubt that the general law suggested by the special cases heretofore observed is correct; the general law is almost proved by *induction*. The mathematician expresses himself with more reserve although fundamentally, of course, he thinks in the same fashion. He would say that the following theorem is strongly suggested by induction:

*The sum of the first* n *cubes is a square.*

(Polya, 1973, p.114-121)

Cases for $n$ = 1, 2, 3 and 5 confirmed the pattern and, as Polya explained, one could inductively claim more generally that "the sum of the first $n$ cubes is a square". Of course, one could have checked more cases, perhaps millions of more cases, to strengthen the claim. Induction suggests that based on the observed evidence accumulated it should be true that the sum of the first $n$ cubes is a square.

Then Polya goes on and formulates the conjecture more precisely. That is, the sum of the first $n$ cubes is not only a square number but also the square of the sum of the first $n$ numbers. He also explains that we were led to this conjecture by induction and that mathematics *in the making* is an inductive science. As he says:

*It is, for* n = 1, 2, 3, . . .

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

3. The law we just stated was found by induction, and the manner in which it was found conveys to us an idea about induction which is necessarily one-sided and imperfect but not distorted. Induction tries to find regularity and coherence behind the observations. Its most conspicuous instruments are generalization, specialization, analogy. Tentative generalization starts from an effort to understand the observed facts; it is based on analogy, and tested by further special cases.

We refrain from further remarks on the subject of induction about which there is wide disagreement among philosophers. But it should be added that many mathe-

> matical results were found by induction first and proved later. Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science.
>
> 4. In mathematics as in the physical sciences we may use observation and induction to discover general laws. But there is a difference. In the physical sciences, there is no higher authority than observation and induction but in mathematics there is such an authority: rigorous proof.
>
> After having worked a while experimentally it may be good to change our point of view. Let us be strict. We have discovered an interesting result but the reasoning that led to it was merely plausible, experimental, provisional, heuristic; let us try to establish it definitively by a rigorous proof.
>
> (Polya 1973, pp. 114–121).

Induction is important because it helps finding patterns among the data, build conjectures and generalize. It lets one "see" the general law before one attempts to prove it. General laws come from somewhere, from accumulating evidence, and do not simply magically appear. Induction provides evidence why something could be true and occasionally the evidence help in establishing that something is true. For example, the evidence initially showed that $1^3 + 2^3 +\ldots+ n^3$ is a square. That is $1^3 + 2^3 +\ldots+ n^3 = k^2$, for some $k$. But, what $k$? Any $k$? Going back to the special cases we experimented with, i.e., $1^3 = 1^2$, $1^3 + 2^3 = 3^2$, $1^3 + 2^3 + 3^3 = 6^2$, $1^3 + 2^3 + 3^3 + 4^3= 10^2$, one could guess a variety of options for $k$. For example, one could try $k = 2n - 1$ (which fails for $n = 3$) or other patterns until one notices that $k$ must actually be $k = 1+ 2 +\ldots+ n$, as Polya says above, or even better that $k = \frac{n(n+1)}{2}$.[10] Then, one knows more precisely and in more detail what needs to be proved. Knowing that $1^3 + 2^3 +\ldots+ n^3$ must equal the specific square $\left(\frac{n(n+1)}{2}\right)^2$ and not just any square $k^2$ facilitates the proof which, as Polya says, must be rigorous. The proof is achieved by mathematical induction,[11] which is based on deductive logic.

Finally, notice that Polya's language and methodology is surprisingly Epicurean, even though the Epicureans unfortunately never produced any mathematics and basically dismissed the subject. Notice that he also speaks of "accumulating further experimental evidence", "analogy" and "observed facts" to enable proofs, which are the typical Epicurean terms of "επιμαρτύρησης" (verification), "κατ' αναλογια τρόπος" (analogical way) and "πρόδηλα" (obvious/observed facts). In conclusion, induction is important in mathematical practice to observe patterns, form conjectures and help to bring about deductive proofs.

## Atomism in Mathematics

The Epicurean critique[12] of Euclidean geometry opened up the way for: (a) a more skeptical stance towards geometry and for things to be revised. Centuries later, non-Euclidean geometries had to be developed in order to facilitate important new physical theories (e.g., relativity theory), (b) a more pragmatic understanding of mathematics, free from metaphysical significations, and considered mainly for its utilitarian purposes, and (c) atomic elements and minimal quantities (quanta), and their discrete properties, to be taken more seriously. Centuries later, quanta came to characterize subatomic particles and more appropriate mathematics had to be employed in order to model such phenomena (e.g., quantum mechanics).

Surprisingly, Epicureans are rarely mentioned in relation to some of the above. In regards to (c) above, and as we mentioned already in the introduction, atomism was a primary motive behind the criticism and rejection of geometry. As Sedley informs us:

> […] that Epicurus believed in a minimal unit of measure out of which not only atoms but also all larger lengths, areas, and volumes are composed, is nowadays widely accepted; and most would also agree that it is not merely a physical minimum, contingent upon the nature of matter, but a theoretical minimum, than which nothing smaller is conceivable. Others both before and since Epicurus have been seduced by similar theories without being led to reject conventional geometry. Yet this is precisely the penalty which a theory of minimal parts should carry with it, for one of its consequences is to make all lines integral multiples of a single length and therefore commensurable with each other, whereas the incommensurability of lines in geometrical figures had been recognized by Greek mathematicians since the 5th ce. Moreover, the principle of infinite divisibility lay at the heart of the geometrical method commonly called the 'method of exhaustion', which was fruitfully developed by Eudoxus in the 4th ce. (Sedley 1976, p. 23).

Sedley (1976), Mau (1973), and White (1989) draw their arguments primarily from the following evidence: Epicurus' phrase "the minimal part in the atom" in his 'Letter to Herodotus', which they understand as some sort of even smaller indivisible minima inside the atoms. Specifically:

> One must believe that the minimal part in the atom also stands in this relation. It is obvious that it is only in its smallness that it differs from what is observed in the case of perception, but it does stand in the same relation. For indeed it is because of this relation that we have already asserted that the atom has magnitude, and have merely extended it far beyond (perceptible things) in smallness. And again we must believe that the minimal and indivisible parts are limits which provide from themselves as primary (units) a standard of measurement for the lengths of larger and smaller (atoms), when we contemplate invisible things with reason. For what they have in common with things which do not permit of movement (across themselves) is

> enough to get us this far; but it is not possible for these (minimal parts) to possess motion and so move together (into compounds) (Inwood and Gerson 1988, p. 10).

Clearly, Sedley's view is based on the passage/evidence above which he interprets as evidence suggesting mathematical atomism, and not just physical atomism. This interpretation might have some good grounds, as Epicurus is quite unclear on the nature of his suggested minima.

Nevertheless, according to our available sources, the Epicureans did not develop any atomistic mathematics.[13] Surprisingly though, special kinds of mathematical atomisms are in some ways implicit in some mathematical theories.[14] We give two examples:

**Example 1** (*Number Theory*). Any natural number $n$ ( $n \neq 1$ ) is the unique product of primes to some powers. (this is known as the *Fundamental Theorem of Arithmetic*). For example:

$$6 = 2^1 \cdot 3^1 \text{ , } 7 = 7^1 \text{ , } 12 = 2^2 \cdot 3^1 \text{ , } 405 = 3^4 \cdot 5^1 \text{ , } 7007 = 7^2 \cdot 11^1 \cdot 13^1$$

The primes form in a way the building blocks of natural numbers, similar to the way atoms are for physical objects. Several of the most important questions in mathematics relate to primes. Primes have several applications in real life, such as cyber-security, speech recognition, etc.

**Example 2** (*Linear Algebra*). Any vector space $V$ has a basis. (This is known as the *Basis Theorem*). That is, $V$ has some elements that in a way 'compose' all other elements of $V$.[15] For example:

$$(3,1,-2) = 3e_1 + 1e_2 + (-2)e_3 \text{ , where } e_1 = (1,0,0),\ e_2 = (0,1,0) \text{ and } e_3 = (0,0,1)$$

Actually, any vector $u = (x,y,z)$ of $R^3$ is a unique linear combination of the vectors $e_1$, $e_2$ and $e_3$. The basis vectors $e_1$, $e_2$ and $e_3$ are in way the building blocks of $R^3$ because each $e_i$ does not reduced to the $e_i$'s (i.e., the $e_i$'s are independent) but all other vectors reduce to them. Orthonormal bases are used in applications such as image processing, quantum mechanics, etc.

Epistemologically speaking, the mathematical atomisms exemplified above allow one to go from the specific to the general in terms of proof.[16] This reminds one of the inductive method we spoke about in Section 3. But, conceptually speaking, how much the examples above relate to Epicurus' mathematical atomism as expressed in passage from the 'Letter to Herodotus'? Actually, Example 1 does not relate much to Epicurus' minima which are to be understood as "a standard of measurement for the lengths of larger and smaller (atoms)" (Inwood and Gerson 1988, p. 10). If $n$ is understood as an object and primes $p_1$, $p_2 \ldots p_n$ as its atoms, then the atoms can grow really large. Even larger than some objects. If $n$ is understood as an atom and primes $p_1, p_2 \ldots p_n$ as its minima, then the minima cannot serve as units to measure atoms because different atoms have different minima and some minima are larger than atoms. Example 2 could perhaps relate to Epicurus' minima a bit better. One could imagine an atom as a pixel in $R^3$ (i.e., a tiny cube) expressed with respect to some basis $e_1$, $e_2$ and $e_3$.

The Epicurean minima could then be the edges of the cube spanned by the $e_i$'s. The minima (i.e., edges) determine the volume of the atom (i.e. pixel cube) which is a standard to measure and compare atoms. Such 3-d pixel atom cannot split into smaller atoms and many such atoms can combine to form larger objects similar to the way many 2-d pixels form images in a computer screen.

The examples above are by no means proposed as a model of Epicurean mathematical atomism. We simply point out some conceptual similarities of the Epicurean mathematical atomism with some types of mathematical atomism that we have today. Once again, it is important to emphasize that we do not claim that the Epicureans held the view that theoretical atomism appears sometimes in mathematics or that it is important, neither we claim that the examples above constitute even a correct explanation of Epicurus' mathematical atomism. Rather, we say that the Epicureans had the relevant conceptual resources (e.g., primary units, minima, etc.) to construct atomistic mathematical models had they known the relevant mathematics and had they been interested in doing so. On the other hand, one wonders if mathematical theories as the ones above, both of a considerable level of abstraction, would have been accepted by the Epicureans (then or now)[17], even as a tool, considering their resemblance to the atomic theory and their several applications in real life.

## Conclusions

The Epicurean contribution to mathematics, direct or indirect, could be summarized as follows:

1. Their underlying logic and empiricism expressed a skepticism towards geometry which led to further expansion of mathematics, and in turn of physics.
2. Their critique of formal logic and their detest of abstraction is felt even today, as 'informal logic' is what characterizes mathematics (in practice[18] and in education).
3. Epicurean logic, even though incomplete (because of mathematics), contains elements of later modern and more developed logics such as Mill's empirical logic, Pierce's semiotic logic relevance logic and fuzzy logic.
4. Their mathematical atomism, even though undeveloped by them, taught and gave ideas to other scientists later to develop theories that better model nature. Even in mathematics, some seek of the most elemental building blocks of things such as primes, vector bases, sets, etc.
5. Epicurean logic is not sufficient to do mathematics, and Epicureans did not produce any mathematics, yet inductive logic seems essential in doing mathematics and some aspects in mathematical proofs seem certainly Epicurean.

The Epicureans were certainly controversial in some of their views. In mathematics, paradoxically, even though the Epicureans dismissed the subject they can nonetheless be found contributing to mathematics indirectly in two ways: Inductive inference helps in deductive mathematical proofs by establishing hypotheses and conjectures and mathematical atomism appears in some ways in mathematics and it is also important and useful. Epicureans did not purposely relate induction to proof or mathematical atomism to mathematics, but they had the relevant conceptual resources (e.g., inductive inference, minima, etc.) to provide a foundation to some aspects of mathematical practice, such as proof, had they been more interested in it.

Epicureans marginalized and discredit mathematics (e.g., Epicurus and Zeno, respectively). Nevertheless, since some important points related to mathematics come out of the Epicurean epistemology, logic and their critique of geometry, then it is reasonable to: (a) Promote those points and connect Epicurean philosophy to some aspects of mathematics, rather than disconnect it. Epicurus dismissed mathematics probably mainly due to ignorance. But what about today? Today we have epistemic and academic reasons to re-assess. (b) Inform modern Epicureans of possible good Epicurean points on mathematics, so some can moderate any dogmatic views they may have or relax any literal attachments to some of Epicurus' now outdated positions. Are today's Epicureans justified in dismissing mathematics?[19] Epicureanism does not have to simply be reduced to a reaction to Platonism or Aristotelianism. (c) Suggest a positive evolution of Epicureanism that strengthens its arguments and make it more relevant to science and life today. Every philosophy admits some evolution of its ideas, even religions.

Most Epicureans insisted on certain modes of thinking that probably caused their dismissal of mathematics which in turn undermined even good things from their philosophy which we see today. Some, like Zeno and Philodemus, begged to differ on some issues. A lesson to all new philosophers in itself.

## Notes

1. This paper is based on a talk entitled "The Epicureans on Mathematics: Some Lessons on Axioms, Logic, Experiment and Proof", given by the author at the '7th Panhellenic Conference on Philosophy of Science' (University of Athens, December 1-3, 2022).
2. The Epicureans perhaps could be justified in a way as neither science nor mathematics were as advanced then as today and they did not have a complete picture. It is possible that today they could have seen mathematics in a quite different way and recognized some of its special aspects.
3. Netz (2015) doubts that there were any Epicurean mathematicians at all. He also claims that the Epicureans were downright hostile towards the profession of mathematics.
4. By "mathematical atomism" we mean the idea that abstract primary objects exist, analogous to the physical atoms, which supposedly form the building blocks of other objects. For example, points compose lines, primes compose integers, etc.

There were different types of mathematical atomisms (i.e., conceptual atomisms) in ancient Greece which are to be distinguished from physical atomisms (i.e., material atomisms). For example, the Pythagoreans suggested the ‘monad’ (a term borrowed later by Leibnitz) which is the indivisible unit that composes all numbers and things. The Platonists suggested the ‘indivisible lines’ which are elemental triangles that compose all solids. Both theories were criticized by Aristotle (Berryman 2022). Epicurean mathematical atomism, based on the Letter to Herodotus (Inwood and Gerson 1988, p. 10), suggests indivisible theoretical minima in the atom which serve as units of measurement of atoms, but are not atoms themselves. An analogue to Epicurean mathematical atomism in order to clarify it is attempted in Section 4.

5. Overall, in ancient Greece, inductive logic was not fully developed. Aristotle discusses arguments from the specific to the general in ‘Posterior Analytics’ but does not provide a full theory. According to Marquand, “Both (Epicurus and Zeno) are occupied with the sign-inference, and look upon inference as proceeding from the known to the unknown. Epicurus, however, sought only by means of hypothesis to explain special phenomena of Nature. Zeno investigated generalizations from experience, with a view to discovering the validity of extending them beyond our experience. This resulted in a theory of induction, which, so far as we know, Epicurus did not possess. In the system of Aristotle, induction was viewed through the forms of syllogism, and its empirical foundation was not held in view. The Epicureans, therefore, were as much opposed to the Aristotelian induction, as they were to the Aristotelian syllogism. It was Zeno who made the first attempt to justify the validity of induction. The record of this attempt will give the treatise of Philodemus a permanent value in the history of inductive logic”. (Marquand 1883, p. 11). Surprisingly, in regards to Epicurus, some even say that Epicurus’ “chief reliance was upon deduction” (DeWitt 1954, pp.7–8).

6. Read (2012, pp. 114–115) says that:

> “The idea that validity requires a relevant connection between premises and conclusion has a long history. It certainly featured in Greek discussion on the nature of conditionals, since Sextus Empiricus, in his history of Pyrrhonism, speaks of ‘those who introduce connection or coherence assert that it is a valid hypothetical whenever the opposite of its consequent contradicts its antecedent’ (Pyrrhoneiae Hypotyposes, ii 111). He does not say who held this view.”

Neither Sextus nor Read mention the Epicureans.

7. For the proof see (Jones 2011). The prime in question is a number bigger than $10^{136}$, much larger than the number of atoms in the observable universe which is estimated to be $10^{80}$. Note also that large primes, usually much larger than the one discussed here, are used in cyber-security.

8. One could perhaps hear this sequence too! In the ‘Online Encyclopedia of Integer Sequences’ (available at: http://oeis.org/) one can also have audio of various sequences, such as the prime numbers, the Fibonacci numbers, etc.

9. Or all humanity's lifetime for that matter. Life on Earth is estimated to come to an end in about 5 billion years (i.e., $5 \cdot 10^9$ years). The number $12\underbrace{11...1}_{136}$, roughly speaking, is way bigger than $10^{100}$ years.
10. Of course, it is well known fact that $1+2+...+n=\frac{n(n+1)}{2}$ and it is proved by mathematical induction. But, proving Polya's claim that $1^3+2^3+...+n^3=(1+2+...+n)^2$ would require knowledge of that very fact where proving that $1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$ would not require it. So, it makes the proof a bit easier if one notices that $1^3 + 2^3 +\ldots+ n^3 = k^2$, where $k=\frac{n(n+1)}{2}$.
11. The proof that $1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$, using mathematical induction, goes as follows:

*Base Step*: For $n = 1$ we have that $1^3=1=\left(\frac{1(1+1)}{2}\right)^2$, so the statement is true for $n = 1$.

*Inductive Step*: We show that if the statement is true for $n$, then the statement is true for $n + 1$. I.e., given that $1^3+2^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$ we show that:

$$1^3+2^3+...+(n+1)^3=\left(\frac{(n+1)(n+2)}{2}\right)^2.$$

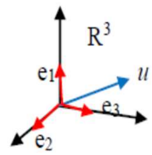
The left-hand side of the latter gives:

$$1^3+2^3+...+(n+1)^3=1^3+2^3+...+n^3+(n+1)^3=\underbrace{1^3+2^3+...+n^3}+(n+1)^3\overset{ind.step}{=}\underbrace{\left(\frac{n(n+1)}{2}\right)^2}+(n+1)^3=$$

$$\frac{n^2(n+1)^2}{4}+(n+1)^3\overset{fact.}{=}(n+1)^2\left(\frac{n^2}{4}+n+1\right)=(n+1)^2\left(\frac{n^2+4n+4}{4}\right)=(n+1)^2\frac{(n+2)^2}{4}=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

12. One of the few systematic critiques. Of course, the Epicureans criticized geometry for their own philosophical reasons but mathematically the critique raised several legitimate points nevertheless, and exposed some of Euclidean geometry's problems.
13. That is until new sources might show otherwise. It is possible that Epicurus' work "On the Angle of the Atom" or one of Philonides' geometric works (written to explain Epicurus' minima (Netz 2015, p. 320 (note #53)) contain atomistic mathematics.
14. Archimedes in "The Method" uses 'indivisibles' to compute areas. Archimedes showed that one can still do mathematics without infinite divisibility (Mau 1973). Yet, he did it heuristically, as after that he still needed rigorous proof.
15. There are, of course, infinitely many bases for a vector space $V$. But, all bases contain the same number of elements. This common number is called the *dimension* of $V$. In the figure below, we see an example of such a basis for the

space $R^3$. The red vectors $e_1$, $e_2$ and $e_3$ form the "natural basis" of $R^3$. For this special basis the $e_1$, $e_2$ and $e_3$ have length 1 and they are pairwise perpendicular to each other. Furthermore, any vector of $R^3$ (e.g., the blue vector $u$ of the figure) is a unique linear combination of the red vectors.




16. Regarding Example 1, one could know when a composite integer $n > 1$ is the product of two primes (i.e., $n = pq$) by simply checking whether the smallest prime factor in its prime decomposition is greater than the cube root of $n$ (Rosen 2011, p. 76). For example, $6 = 2 \cdot 3$ because $2 > \sqrt[3]{2} \approx 1.25$. Numbers that are the product of very large primes are very important in cryptography. Regarding Example 2, one could know how a linear transformation $T$ in $R^3$ transforms vectors in general (i.e., its formula) by simply knowing how it transforms the vectors of the basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. If $T$ transforms $e_1$ to $(1,0,0)$, $e_2$ to $(0,1,0)$ and $e_3$ to $(0,0,0)$, then as $(x,y,z) = xe_1 + ye_2 + ze_3$ one can easily have the full transformation $T(x,y,z) = (x,y,0)$, which is a projection of any vector on the $e_1e_2$-plane.

17. Since the Epicureans were familiar with Euclid's *Elements*, as they criticized geometry, they must have been familiar with Books 7 and 9 of the Elements which deals with prime numbers and in particularly the Fundamental Theorem of Arithmetic. Nevertheless, it seems that the Epicureans neither commented on primes nor exploited the opportunity to argue for some type of mathematical atomism.

18. Whether we actually use or need formal logic in mathematical proofs see Aristidou (2020b).

19. There are several misconceptions among modern Epicureans about the nature and applications of mathematics, especially pure mathematics. Also, there are misconceptions about the principles and scope of logic, especially of non-classical logics such as fuzzy logic, quantum logic, etc. Some typical misconceptions can be found in Patzoglou (2011), Stamatiadou (2013), Altas (2015). For example, a common misconception is that fuzzy mathematics is some alternative mathematics that challenges "classical" mathematics. It is not. Fuzzy mathematics is a branch of mathematics that extends the classical notion of a set by means of classical mathematics and develops tools that enable scientists to model imprecise and fuzzy situations. We are hoping to expand further on such issues in a future paper.

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