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Cartesian Products: From Elements to Maps

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This paper addresses an ambiguity, within Zermelo-Fraenkel set theory, arising from the attempt to define the elements of cartesian products, viz. ordered pairs, and hence an ambiguity in the attempt to define cartesian products themselves. The proposed solution takes us to a different conceptual framework: category theory. More specifically, to the category of sets and maps between them, in which the ambiguity is removed in a natural way. The case of cartesian products is only one example of the interplay between these two ways of axiomatizing the concept of set. The proposed solution to the ambiguity requires, in particular, to make a transition from thinking of elements as a primitive concept (as it is done in Zermelo-Fraenkel set theory), to thinking of them as special maps, which is the basic concept in the category of sets.

Keywords: *category of sets, cartesian products, functions, ordered pairs, Zermelo-Fraenkel set theory.*

Introduction

Set theory and category theory share a remarkable feature: many mathematical concepts from various and diverse areas can be defined within them, although certainly in quite different ways. However, neither theory arose with this purpose. The origins of set theory go back to the 1870's in the work of the German mathematician Georg Cantor. Its first axiomatization, published in 1908, is due to another German mathematician, Ernst Zermelo[1]. In 1964, the American mathematician and philosopher, F.W. Lawvere, proposed for the first time an axiomatization of the category of sets and maps between them[2]. Thirty nine years later Lawvere, in co-authorship with Robert Rosebrugh, published a more thorough axiomatization of this category[3]. All the axioms for this category that I use in this paper, are from this second axiomatization.

The overall aim in the analysis presented here is to show an instance of what in my view is a quite fruitful interplay between these two approaches to the mathematical concept of *set*: More specifically, I focus here on the many different ways there are for defining the concept of *ordered pair* in Zermelo-Fraenkel set theory (hereafter ZF), all of them equally satisfactory. Whichever definition one chooses, it will always be arbitrary, perhaps even just a matter of taste. In other words, there is no strictly mathematical reason for preferring one definition over all the others. So it is in this sense that I consider the concept of *ordered pair* in ZF ambiguous[4]. The aim

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[1]Zermelo E (1908a).
[2]Lawvere FW (1964).
[3]Lawvere FW, Rosebrugh R (2003). To my knowledge, no other axiomatizations of the category of sets have been proposed since then.
[4]In philosophy of language this phenomenon is called *underdetermination of the referent*, where

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of the paper is then to show that by moving from ZF to category theory (and more specifically, to the category of sets and maps between them), we no longer face the problem of having to choose one, among many other possible and equally good ways, for defining the concept of *ordered pair*. In the category of sets and maps between them, this ambiguity simply dissolves and in a natural way.

The paper does not contain, nor does it aim to offer, any new mathematical results. Indeed, its mathematical content is quite basic, not only in the case of the definitions and axioms used, but also in the proofs given herein. The paper is rather an exploration of the interplay between two axiomatic characterizations of the concept of *set*: ZF and the category of sets and maps between them (hereafter CS). The particular interplay here concerns the definition of cartesian products and hence of their elements. Part II shows in detail how the dilemma arises in ZF, and Part III explains what happens when we move to CS.

As far back as 1964, another similar problem was addressed by the American philosopher Paul Benacerraf[5]. Back then, many philosophers had argued that ZF was the right foundation for mathematics, and Benacerraf P (1965) argued precisely against this claim. In any case, the dilemma I address here lends itself to philosophical scrutiny. Moreover, category theory has raised many issues among philosophers of mathematics[6]. So another caveat is in order. Just as this paper does not aim to give new mathematical results, it does not either aim to address philosophical issues, not even the standard ones such as *What are the right foundations for mathematics? What are numbers, sets, functions, etc.? What is the meaning of mathematical statements? What is the nature of mathematical truth?*[7] What it does aim to offer is an introduction, *via* ZF, to a few basic concepts from category theory to anyone interested in it but who may find it too abstract to understand.

Part I: Elements and maps

Our starting point is ZF. In this theory everything is a set and there is only one basic concept, *viz.* that of *element* or *membership*, which is a relation between sets[8].

The following statement is called the axiom of extensionality

Extensionality Axiom

For any given sets A and B, $A = B$ if and only if A and B have exactly the same elements.

In its contrapositive form, this axiom states that $A \neq B$ if and only if there exists an element a in A such that a is not an element of B (or, equivalently, if and only if there exists an element b in B such that b is not an element of A).

the *referent* in this case is precisely "the" ordered pair (a, b).

[5]Benacerraf P (1965).

[6]See, *e.g.* Landry E (2017),

[7]Shapiro S (2000) addresses precisely issues such as these.

[8]Like most authors, I am not considering the binary relation of equality as a formal part of ZF in the way membership is. Following the vast majority of textbooks, I simply assume that it is reflexive, symmetric and transitive, with its meaning intuitively given.

This axiom, like all others in ZF, can be written entirely in logical symbols. However, in many contexts it is interpreted as stating that a set is completely determined by its elements. And this in turn can be interpreted or read in different ways. For example, if we are given some set A and we somehow "remove" an element from it, the resulting set is different from A. Analogously, if we "add" a new element to A, we end up with a set different from A. The interpretation I propose for this axiom[9] is the following: if one claims that a certain set exists, one must state precisely what its elements are—and this could be done in many ways, *e.g.* by giving a recursive definition, by listing them one by one, by the requirement that they satisfy some formula, etc. If one cannot do this, then one is not justified in claiming the existence of the set involved. I shall use this interpretation in Part II below. In ZF, cartesian products, intersections, unions and many other sets, are defined or axiomatized (or even proven to exist) by stating (or exhibiting) what its elements are or are not, as it is the case with the empty set.

There is, however, another important point concerning the requirement that a set be completely determined by its elements. In what is called basic *algebra of sets*, there are many theorems stated as equalities between sets. For example, one theorem states that for any given sets A, B and C, the following equality holds

$$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$$

In order to prove this, one needs to know what the elements of the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cup C)$ exactly are. And the same happens in the case of cartesian products, which will be discussed in Part II below.

One of the first axioms of ZF is somewhat strange for it tells us that there is *a* set that has no elements whatsoever. With these two axioms, one can easily prove that there is only one set without elements. This justifies the use of a special symbol and a name for it: *the* empty set, denoted as $\varnothing$.

The pairing axiom allows us to give life to the set $\{\varnothing, \varnothing\}$ which apparently has two elements. However, it follows directly from the axiom of extensionality that $\{\varnothing, \varnothing\}$ is exactly the same set as $\{\varnothing\}$. Notice that $\{\varnothing\} \neq \varnothing$ because the set $\varnothing$ does not belong to $\varnothing$, since this latter has no elements. In contrast, $\varnothing$ is indeed an element of $\{\varnothing\}$, and hence $\{\varnothing\}$ is *not* empty. Hence $\{\varnothing\} \neq \varnothing$. So now we have two different sets living in the ZF universe: $\varnothing$ and $\{\varnothing\}$.

It is precisely the set $\{\varnothing\}$, together with some special properties it has, what we are going to use to make the transition from thinking of elements in the sense of ZF, to thinking of them in terms of maps. So we now turn our attention to this latter concept.

Within ZF, the concept of *map* or *function* is not basic, it is defined[10] in terms

[9]This proposal is not intended as a mathematical result that follows from the axiom of extensionality. I do not intend it either as a philosophical interpretation, although perhaps that is what it is. My interpretation is inspired in the criticisms to Zermelo´s axiom of choice at the beginning of the XX century. The axiom stated the existence of a certain function, and Zermelo´s critics demanded that he exhibited the function, which not only he couldn't do but, as we now know, it just cannot be done within ZF.

[10]Some would say *reduced* instead of *defined*. Indeed, many mathematical concepts can be so reduced, and before category theory entered the philosophical realm, this is what *foundations of*

of sets (hence in terms of the membership relation). However, for the purpose of this section, we are going to use what one might call the *intuitive* concept of function. This is the concept that is mostly used in the practice of mathematics and in many real life cases.

Let us consider, for example, the following two sets:

$$A = \{a, b, c, w\} \text{ and } B = \{7, 15, 18\}$$

A function *from A to B* should assign, to *each* element of A, *one and only one* element of B. We use the letter f to denote the function in question and, for any element x of A, the expression $f(x)$ shall denote *the* element of B that the function f assigns to the element $x \in A$. The following three are all examples of functions from A to B:

(1) $f(a) = 7, f(b) = 18, f(c) = 18, f(w) = 15$

(2) $f(a) = 7, f(b) = 7, f(c) = 7, f(w) = 7$

(3) $f(a) = 15, f(b) = 7, f(c) = 18, f(w) = 18$

In contrast to the second example, and at least on the surface, there seems to be no rule for the assignments in the first and third examples. Nonetheless, each one establishes a function from A to B. For the second example, we can simply say that the function f is given by $f(x) = 7$ for all $x \in A$. So, as long as each element of A gets associated with one and only one element of B, even if the association is completely arbitrary, there is a function from A to B.

Let us now consider *the* set $\{\varnothing\}$ and take *any arbitrarily chosen* set A. Then clearly, there is one and only one function from A to $\{\varnothing\}$: all elements of A get assigned to the only element of $\{\varnothing\}$, *viz.* the empty set $\varnothing$. In other words, if we call this function f, then $f(x) = \varnothing$ for all $x \in A$. And this holds true even if A is the empty set. This is due to the fact that in ZF functions are defined as certain *subsets* of cartesian products. So a function f from A to B is a collection of ordered pairs (a, b) with $a \in A$ and $b \in B$, in which b, the second coordinate is the only element of B that the function f assigns to $a \in A$. In order to make this more conspicuous, we change the notation and write $(a, f(a))$ to express that this is an element of the function f from A to B, so that $(a, f(a)) \in A \times B$. So, in general, a function f from A to B is (in ZF) a subset of $A \times B$ of the form

$$\{(a, f(a)) \mid a \in A\}$$

with the property that for any $x, y \in A$, if $x = y$, then $f(x) = f(y)$. Or, equivalently, if $f(x) \neq f(y)$, then $x \neq y$.

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mathematics was usually taken to mean. A cursory look at the contents of a few randomly chosen introductory textbooks on set theory exemplifies this: following the definition of the natural numbers, usually comes that of the integers, then that of the rationals, the reals, etc.

With this concept of function, it is clear that $\varnothing \subseteq \varnothing\times\{\varnothing\}$. However, in order for the empty set to *not* be a function, the above mentioned property must fail. This means that there must be at least one element of $\varnothing$ that gets assigned to *two different elements* of $\{\varnothing\}$, which is clearly impossible. So, strange as it is, the formal concept of function in ZF, together with the fact that the empty set is a subset of *any* set, imply two things: first, that $\varnothing$ is also a set of ordered pairs (albeit empty); and, secondly, that $\varnothing$ is a function (albeit empty as well), in this case from $\varnothing$ to $\{\varnothing\}$. To sum up, given any set A, including the empty set, there is always one and only one function from A to the one-element set $\{\varnothing\}$.

Here is a key point about the above arguments: instead of $\{\varnothing\}$ we could have taken *any* other set as long as it had just one element. And this is the starting point of our transition from thinking of elements as a (basic) relation between sets to thinking of them as special maps. So let us denote an arbitrarily given one-element set as 1. Thus 1 has the property that for given any set A, including the empty set, there is always *one and only one function* from A to 1. We shall denote the uniqueness of functions with dotted arrows as follows, for any given sets A and B

Diagram 1

$$A \overset{f}{\dashrightarrow} B$$

or

$$f\colon A \dashrightarrow B$$

Let now A be any given set and consider all functions *from* 1 *to* A

Diagram 2

$$1 \longrightarrow A$$

In this case, the number of such functions depends of course on the number of elements that A has. The set of all functions from 1 to A is denoted as A^1. More generally, for any given sets A and B, the set of all functions from A to B is denoted as B^A. And there is a good reason for this. When the exponent is 1, it is not difficult to see that there is a bijection between A^1 and A. And when A is empty, A^1 is also empty. In terms of cardinalities, this case yields the familiar equation $0^1 = 0$. Another familiar equation arises when A has only one element: $1^1 = 1$. But we can say more about this latter case. Since for any one-element set A, the set A^1 of functions from 1 to A has only one element, this element must be the identity function on A. So we see there is a very close connection between the arithmetical operation of exponentiation, and the cardinality of sets of functions B^A. Indeed, in general $\mathrm{card}(B^A) = \mathrm{card}(B)^{\mathrm{card}(A)}$. Thus if A is the empty set $\varnothing$ and the cardinality of B is n then the cardinality of $B^\varnothing$ is n^0, and this number is always equal to one. For the empty set ∅ we have that $0^0 = 1$. This equation tells us that there is exactly one function from ∅ to ∅ (albeit empty within ZF), which we shall call *the* identity function on ∅. For any non-empty set B with cardinality n, the equation $n^0 = 1$ tells us that there is exactly one function from ∅ to B. And we shall call it *the inclusion function* of ∅ into B, which is accordance with the theorem in ZF stating that the empty set is a subset of *any* set. Finally, for any set A, including ∅, the set A^A is never empty, for there always exists *the* identity function on A. In arithmetical terms, for any natural number n, including 0, $n^n \neq 0$.

But what about functions *from* 1 *to* any given set A? If A is empty, the number of these functions is 0^1 which is equal to 0. In other words, the number of these functions is equal to the number of elements of A. Suppose now that A is not empty. In this case we can always define functions $1 \to A$ as follows. Let $a \in A$ be an arbitrary element of A and let • denote the only element of 1. So we define $F_a(\bullet) = a$. And we can do this for *each element of* A. In other words, every element a of A determines a function $F_a : 1 \to A$.

Let us now consider an arbitrary function G: $1 \to A$. Then G determines a unique element of A, namely, $G(\bullet) \in A$. It is not difficult to show that this way of "picking out" elements of A through functions G: $1 \to A$, together with the definition above of F_a for each $a \in A$, establish a bijection between A^1 and A itself. In terms of cardinalities, if n is the number of elements of A, this yields the familiar equation $n^1 = n$ (which, as we just saw, includes the case when $n = 0$). Whenever a bijection exists between sets, we say that they are *isomorphic*. If we denote the set of all functions from 1 to A as A^1, what we have is then A and A^1 are isomorphic sets. But we leave this important point for now[11], in order to take a first look at the world of categories.

In category theory one fundamental basic notion is that of *morphism*. Categories are constituted by two "ingredients": *objects* and *morphisms*. And any given morphism always comes with two associated objects: its *domain* and its *codomain*. The notation is the same as the one we use for functions in general: morphisms are usually denoted with the letters f, g, h… and their domains and codomains with the letters A, B, C,… We can think of the axioms characterizing the concept of *category* as stating well-known properties about functions, properties that in ZF are theorems.

Following mathematical usage, arrows are used to denote morphisms. So if f is a morphism with domain A and codomain B, we write

Diagram 3.

$$A \xrightarrow{\;f\;} B$$

or

$$f : A \longrightarrow B$$

[11]See the end of this section and Part IV below. It is in this latter, where I propose to use this bijection to make a transition from thinking of elements in the sense of ZF to thinking of elements in the sense of the category of sets.

So here is the axiomatic characterization of the concept of *category*.

A *category* ***C*** is given by its *maps* and *objects*.

Let then $f: A \longrightarrow B$, $g: B \longrightarrow C$ and $h: C \longrightarrow D$ be any given objects and morphisms of ***C***.

Axiom 1: *Existence of composition of any pair of morphisms in which the codomain of one coincides with the domain of the other*

So there exists a morphism $gf: A \longrightarrow C$ (and also $hg: B \longrightarrow D$)

Axiom 2: *Composition is associative*

So $h(gf) = (hg)f: A \longrightarrow D$

Axiom 3: *For any object A of* ***C****, there is a special morphism with domain and codomain A, and which is neutral with respect to composition. This morphism is called the identity on A*[12] and it is denoted as $id_A: A \longrightarrow A$.

So given any objects A, B and C and any morphisms $f: A \longrightarrow B$ and $g: B \longrightarrow C$, the following two equalities hold

$$(id_B)f = f \quad \text{and} \quad g(id_B) = g$$

And that is all. Of course, particular categories, or even types of categories, like the category of sets and map, have additional axioms. The first one for CS is the following:

Axiom 1 (CS)

There is *a* set, say X, with the property that for any given set A there is always exactly one map from A to X.

Notice that the axiom characterizes *X not* by stating what its elements are, but by telling us how it relates to other sets, including X itself, through *maps*[13]. It is in this sense that this set X resembles one-element sets in ZF, for as we saw earlier, given any one-element set 1 and any arbitrary set A, there is exactly one function from A to 1. Now, in ZF there is an infinite number of one-element sets, so each one of them has this latter property. One may then wonder whether in CS there is also more than one set satisfying Axiom 1 above. The answer is that the theory simply does not tell us how many there are or might be. What it does tell us is that this question does not matter and this is why[14]. Let us suppose there is another object Y also satisfying Axiom 1 above. Hence there is a unique map f from Y to *X and* a unique map g from *X to Y*

[12]It is called *the* identity morphism because it is unique. However, the uniqueness is not part of the axiom since it follows directly from Axioms 1 and 3.

[13]This is typical in category theory. When an object such as X is characterized by postulating the existence of unique map(s) from other object(s) to X or from X to other object(s), we say that X has been characterized by a *universal property*. This has a very important consequence as we will see in the case of Axiom 1.

[14]This consequence is of the utmost importance, and it follows every time an object is characterized by a universal property.

Diagram 4

$f\!:Y \dashrightarrow X$

and

$g\!:X \dashrightarrow Y$

By composing these maps with each other, we obtain the maps *gf* and *fg*

Diagram 5

$$Y \xrightarrow{gf} Y$$

and

$$X \xrightarrow{fg} X$$

However, since CS is a category, there are also the identity maps

$$id_Y\!: Y \longrightarrow Y \quad \text{and} \quad id_X\!: X \longrightarrow X$$

But our hypotheses are that both *X* and *Y* satisfy Axiom 1. Therefore, $gf\!: Y \longrightarrow Y$ must be the same map as id_Y and $fg\!: X \longrightarrow X$ must be the same map the identity on *X*, id_X. We then say that *X* and *Y* are *isomorphic*. And the overall conclusion is that any object satisfying Axiom 1 (CS) is *unique up to isomorphism*. And that is all that CS can say about the question concerning the number that there might be of sets satisfying this axiom. And this is alright, for it is all that is needed in CS (and also in category theory in general).

Objects satisfying Axiom 1 above are called *terminal*. So this axiom is guaranteeing that in CS there is at least one terminal object. We shall denote as 1 for reasons we will see shortly. The defining property of 1 is then that for any given set *A* there is a unique map from *A* to 1

$A \dashrightarrow 1$

The first point to notice is that the identity map on 1 is the only map from 1 *to* 1. Let us now go back to the discussion within ZF in which we saw that for any given set *A*, there is a bijection between the set of functions A^1 and *A* itself. This bijection tells us in particular that the sets A^1 and *A* have the same number of elements. But I propose to see it as telling us that there is a close connection between two *concepts*: on the one hand, that of a *function from* any one-element set *to* any given set *A*; and, on the other hand, the concept of *element*. As we have just seen, in

CS there is just one map from 1 to 1. Hence the notation we chose.

Part II: Cartesian Products and their Elements

In ZF everything is a set. So the elements of a sets are also sets, whose elements are in turn sets as well, whose elements are also sets, and so on.

The empty set axiom gives us *the* set $\varnothing$, the pairing axiom the set $\{\varnothing\}$. If we apply the pairing axiom again, we can obtain many other sets, such as the following three:

$$\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}, \{\{\{\varnothing\}\}, \{\varnothing\}\}$$

Thus in ZF, not only are cartesian products sets, but also their elements. Let us see how this comes about.

Let A and B be arbitrary sets in ZF. Their *product* is denoted as $A\times B$ and it is defined as a certain subset of the power set of the power set of the union $A\cup B$. In symbols

$$A\times B \subseteq \mathcal{P}(\mathcal{P}(A\cup B))$$

The elements of $A\times B$ are called *ordered pairs*, and they are denoted as (a, b) with $a\in A$ and $b\in B$. Since $\mathcal{P}(\mathcal{P}(A\cup B))$ is a set, then all its elements are sets as well. And since all the elements of $A\times B$ are also elements of $\mathcal{P}(\mathcal{P}(A\cup B))$, then all elements of $A\times B$ are elements of $\mathcal{P}(\mathcal{P}(A\cup B))$ as well. Hence, all elements of $A\times B$ are sets.

Since an element (a, b) of $A\times B$ is in addition an element of $\mathcal{P}(\mathcal{P}(A\cup B))$, one needs to make sure that whichever definition for (a, b) one chooses, the resulting *set* (a, b) does indeed belong to $\mathcal{P}(\mathcal{P}(A\cup B))$. In addition, in order for (a, b) to be an *ordered* pair, its definition must distinguish (a, b) from (b, a) whenever $a \neq b$. For example, the axiom of extensionality implies that

$$\{a, b\} = \{b, a\}$$

So whenever $a \neq b$, we cannot choose the set $\{a, b\}$ as our definition for (a, b). Indeed, the requirement that guarantees that these sets (a, b) are indeed *ordered* is the following, where $a, a'\in A$ and $b, b'\in B$

$$(a, b) = (a', b') \text{ if and only if } a = a' \text{ and } b = b'$$

We shall refer to this requirement for any satisfactory definition of *ordered pair*, as simply ROP. Recall that according to the axiom of extensionality, any set is completely determined by its elements. As a consequence, if we want to define or construct the set $A\times B$ we must state clearly what its elements are. This in turn forces us to state clearly what the elements of each ordered pair (a, b) are. But whichever definition we may propose, we must verify that it satisfies ROP.

We have finally come to the problem or dilemma in ZF about the definition of ordered pairs: there are many ways of defining (a, b) and ZF has no means for telling us which one is *the* right one. The grounds for preferring one definition over others, will always be non-mathematical[15]. Here are three proposals, all of them satisfying ROP, that have been made

$(a, b) = \{\{ \{a\}, \varnothing\}, \{\{b\}\} \}$ (Norbert Wiener, 1914)

$(a, b) = \{ \{a, \varnothing\}, \{b, \{\varnothing\}\} \}$ (Felix Hausdorff, 1914)

$(a, b) = \{ \{a\}, \{a, b\} \}$ (Kazimierz Kuratowski, 1921)

In Kuratowski´s definition, when $a = b$, the set (a, b) has just one element. It might then be a little bit surprising that the definition does satisfies ROP. Moreover, the proof is straightforward, albeit somewhat tedious.[16]

In contrast, even when $a = b$, in the first two proposals, the sets (a, b) have exactly two elements. In other words, even when $a = b$, the following inequalities hold

$\{\{a\}, \varnothing\} \neq \{\{b\}\}$

and

$\{a, \varnothing\} \neq \{b, \{\varnothing\}\}$

It is Hausdorff's definition the one that in my view is the most interesting. What makes the difference between the sets $\{a, \varnothing\}$ and $\{b, \{\varnothing\}\}$ is that $\varnothing$ is empty whereas $\{\varnothing\}$ is clearly not. But that is all. So one might as well choose *any different* sets X and Y and define

$$(a, b) = \{ \{a, X\}, \{b, Y\} \}$$

It is indeed surprising that this definition also satisfies ROP.[17] What I find of

[15]If we think of the elements of a cartesian product $A\times B$ as some kind of objects, then this multiplicity of options for being "the" right choice for (a, b) can be described as an undetermination of the referent of (a, b). What I show in Part IV below is that in the category of sets the of (a, b) is completely determined.

[16]It is perhaps because it does not introduce any other sets besides a and b, that Kuratowski's definition is the one most commonly used. Enderton HB (1977), p.28 after having introduced Kuratowski's as his chosen definition of an *ordered pair* and having proved that it does satisfies ROP, he comments: "As you have probably observed, our decision to use the Kuratowski's definition […] is somewhat arbitrary. There are other definitions that would serve as well. The essential fact is that satisfactory ways exist of defining ordered in terms of other concepts of set theory." I would add that Enderton's decision is most definitely arbitrary and that what he calls "the essential fact" is a problem within ZF, a problem which, as I hope to show in this paper, has a satisfactory solution in category theory. Of course, if one is just thinking of ZF as a suitable foundation for mathematics, it might in fact be enough that *ordered pair* can be defined within ZF.

[17]Indeed, this example appears as an exercise in Hrbacek K, Jech T (1999), p.18 and it consists

special interest is that this definition opens up an infinite number of possibilities for defining ordered pairs in ZF. And this is what I have called the problem or dilemma concerning the definition of the elements of cartesian products. So it is due time that we see how, as I claim, this dilemma vanishes in category theory. More specifically, in the category of sets and maps.

Part III: Products in Category Theory

As with most concepts in category theory, products of objects are defined by what are called *universal properties*[18]. So let ***C*** be a given category and let X and Y be any two given objects in ***C***. *A* product for X and Y in ***C*** consists of the following:

(1) an object in ***C***, which we shall denote as $X{\times}Y$, together with two morphisms in ***C***, called the *projections*

Diagram 6

$$X \xleftarrow{\;p_X\;} X \times Y \xrightarrow{\;p_Y\;} Y$$

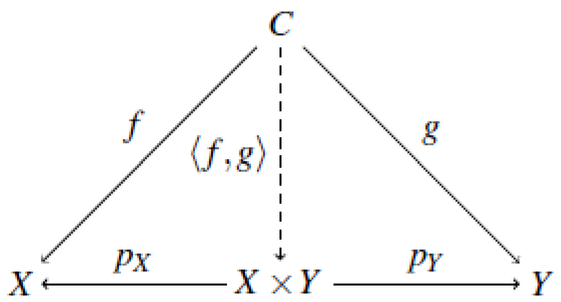
(2) *and* the following *universal property*

for any object C and any maps $f\colon C \longrightarrow X$ and $g\colon C \longrightarrow Y$ in ***C***, there is a *unique map* in ***C***, denoted as $<f, g>\colon C \longrightarrow X{\times}Y$ since it depends on both f and g, such that $p_X<f, g> = f$ and $p_Y<f, g> = g$. Another way of expressing this is by saying that the following *diagram commutes*

in stating and proving the corresponding analogue of ROP above.

[18] Although I prefer the word *limiting* instead of *universal*, I will follow standard terminology. The reason I find *limiting* a better adjective for these properties is that all of them are particular cases of the concept of *limit* in category theory. It can be argued that terminal objects, products and many other concepts defined by universal properties are "the best" approximations to various types of diagrams, where a *diagram* has also a formal and general definition that includes the various types of diagrams I discuss in this paper. But developing in detail such an argument falls far beyond the scope of this paper, so I adhere to standard terminology.

Diagram 7



Recall that Axiom 1 (CS) characterized the set 1 by postulating, for any set *X*, the existence of a *unique map* from any set *X to* 1. In the case of a product for *X* and *Y*, such as *X*×*Y* (together with the two projections), the definition also asserts the existence of a *unique map* from any set *C to X*×*Y*, although in this case such map must satisfy some additional properties (expressed by the commutativity of the diagram above). We can think of the corresponding so-called universal properties in the following sense.

In the case of the set 1, we can think of it as a diagram or figure consisting of just one point and no maps at all:

$$\bullet$$

The universal property is thus telling us that any other diagram also consisting of just one point

$$\blacklozenge$$

always bears a special relation to • *via* a *unique map* from ♦ *to* • . There are no further conditions since the diagram • contains no maps, it consists of just one point. Now, the general, abstract diagram corresponding to products is a little bit more complicated, for it starts with three objects and two maps

Diagram 8

$$\bullet \longleftarrow \nabla \longrightarrow \blacklozenge$$

Thus, the universal property of a product for any given objects *X* and *Y*

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$$

then tells us that *any* similar diagram such as Diagram 8 above, bears a special relation to the product diagram, again *via* a unique map from ∇ *to X*×*Y*, but in this case such unique map must satisfy some further conditions, due to the more complex "structure" of the product diagram. These further conditions are expressed precisely in the universal property of binary products (item (2) above).

As it happened in the case of the terminal object 1 in CS, products too are unique up to isomorphism. To see how this comes about for products, consider again *a* product for X and Y

Diagram 9

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$$

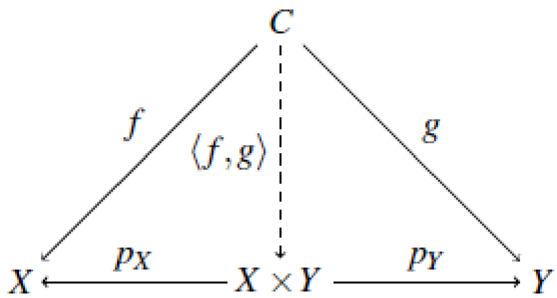
and suppose that the following diagram is also a product for X and Y

Diagram 10

$$X \xleftarrow{f} C \xrightarrow{g} Y$$

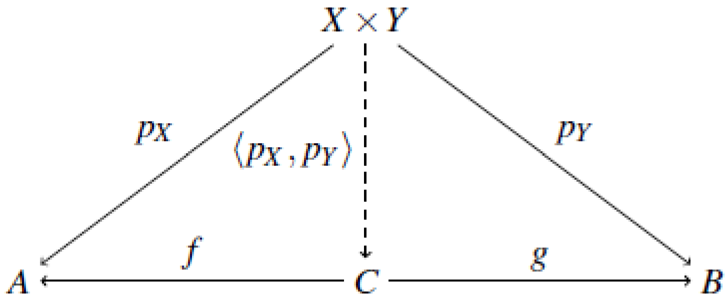
Then, due to the universal property of $X \times Y$, together with its two corresponding projections, there exists a unique map $\langle f, g\rangle: C \longrightarrow X \times Y$ making the following diagram commute

Diagram 11



Analogously, due to the universal property of C together with the maps f and g, there exists a unique map $\langle p_X, p_Y\rangle: X \times Y \longrightarrow C$ such that the diagram below commutes

Diagram 12



We can now compose the maps

$$\langle f, g\rangle : C \longrightarrow X \times Y$$

and

$$\langle p_X, p_Y\rangle : X \times Y \longrightarrow C$$

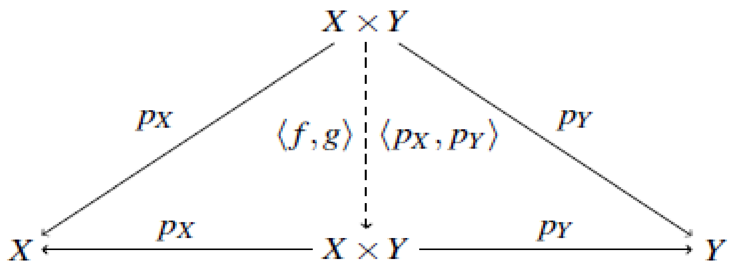
and obtain two more maps

$$<f, g><p_X , p_Y>: X\times Y \longrightarrow X\times Y \quad \text{and}$$

$$<p_X , p_Y><f, g>: C \longrightarrow C$$

Let us now consider the composition $<f, g><p_X , p_Y>: X\times Y \longrightarrow X\times Y$ and the following diagram

Diagram 13



We now compose this map $<f, g><p_X , p_Y>: X\times Y \rightarrow X\times Y$ with the first projection p_X : $X\times Y \rightarrow X$, but keeping in mind our hypotheses (see the left hand side triangles in Diagrams 11 and 12 above) about the map

$$<f, g>: C \rightarrow X\times Y$$

Since composition is associative, we have the following equalities:

$$p_X\,(\langle f, g\rangle\langle p_X\,, p_Y\rangle) = (p_X\langle f, g\rangle)\langle p_X\,, p_Y\rangle = f\langle p_X\,, p_Y\rangle = p_X.$$

Therefore, the left hand side triangle in Diagram 13 is commutative. Using the commutativity of the corresponding right hand triangles from Diagrams 11 and 12, one can easily show that the right hand triangle in Diagram 13 is also commutative:

$$p_Y(\langle f, g\rangle\langle p_X\,, p_Y\rangle) = (p_X\langle f, g\rangle)\langle p_X\,, p_Y\rangle = gf\langle p_X\,, p_Y\rangle = p_Y.$$

So the composite map $\langle f, g\rangle\langle p_X\,, p_Y\rangle: X\times Y \to X\times Y$ makes commutative both triangles in Diagram 13. But clearly, the identity map on $X\times Y$

$$id_{X\times Y}: X\times Y \to X\times Y$$

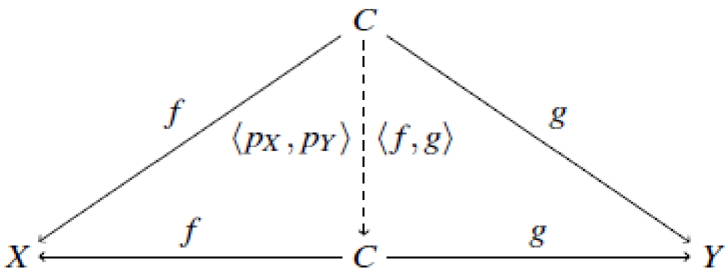
also makes the whole of Diagram 13 commutative, and since the map going from $X\times Y$ to $X\times Y$ in this diagram is unique, it follows that

$$\langle f, g\rangle\langle p_X\,, p_Y\rangle = id_{X\times Y}: X\times Y \to X\times Y.$$

In Diagram 14 below, the map going from C to C is the composition of the two maps

$$\langle f, g\rangle: C \to X\times Y \quad \text{and} \quad \langle p_X\,, p_Y\rangle: X\times Y \to C$$

Diagram 14



Using again our hypotheses from diagrams 11 and 12, together with the axiom stating that composition is associative, we can argue in a completely analogous way as we did for showing that

$$\langle f, g\rangle\langle p_X\,, p_Y\rangle = id_{X\times Y}: X\times Y \to X\times Y$$

and in this case conclude that

$$<p_X, p_Y><f, g> = id_C$$

So products, like terminal objects such as 1, are unique up to isomorphism. In the following section we will see an important consequence this has in the context of defining ordered pairs within ZF, or perhaps more accurately, how it helps us solve the dilemma arising from having to choose one among the infinite number of possible ways in ZF to define the set (a, b), for any given sets a and b.

Part IV: Elements of Products in CS

Recall that in Part I we saw that, in ZF, for any given set A and any one-element set 1, there is a bijection between the set of functions *from* 1 *to A* and the elements of A. This means that the set of these functions, which is denoted as A^1 is isomorphic to A. I propose to go further and think of this bijection as establishing a (bijective!) correspondence between the *concept* of *element* as a basic, undefined notion, and the derived concept—all this in ZF— of *function*. In other words, I propose to see this bijection as one between two concepts: that of *element* of a given set A and the concept of *function from* 1 *to A*. I propose to use this conceptual bijection to transit from thinking of the *elements* of a set A as a basic, intuitively understood concept, to thinking of them as *functions* from any one-element set to A. Once this is achieved, we can then enter the world of sets as these are axiomatized in CS. For in CS *map* is a basic, undefined concept, and the concept of *element* is defined in terms of it. More precisely, given any set A in CS, an element of A is defined as a *map* from 1 to A. In particular, and as we saw earlier, the identity on 1 is then an element of 1. In fact, it is its only element. So in CS, 1 is a one-element set.

Let us finally see how, with this conceptual change, we can solve the problem concerning the definition of the elements of cartesian products, that is to say, the problem of defining ordered pairs. And for this we need the following

Axiom 4 (CS)

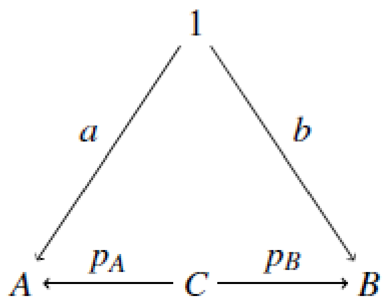
Any given sets A and B have a product.

So let then A and B be any given sets in CS and consider a product for them

$$A \xleftarrow{\;p_A\;} C \xrightarrow{\;p_B\;} B$$

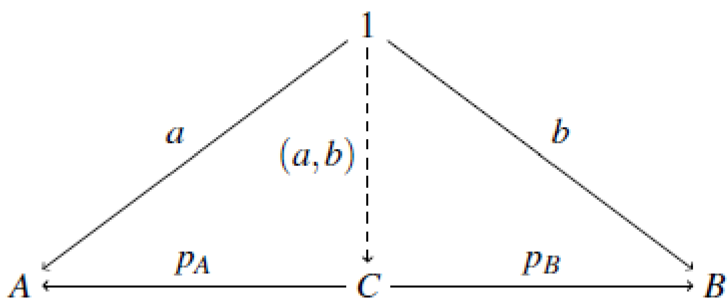
Consider now an arbitrary element of A $a: 1 \longrightarrow A$ and an arbitrary element $b: 1 \longrightarrow B$ of B. We now have the following

Diagram 15



By the universal property of products, there is a unique map from 1 to $A \times B$—which we might as well now denote as (a, b)—and hence a *unique element* of $A \times B$, such that the following diagram commutes

Diagram 16

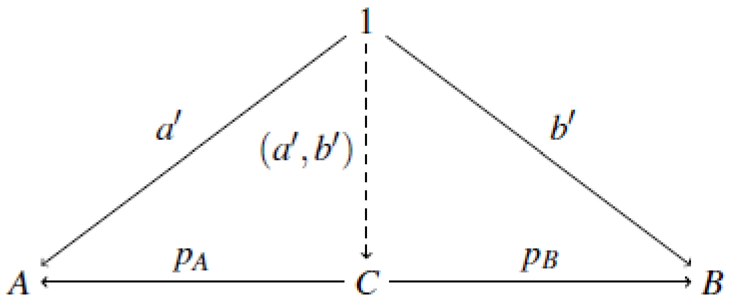


The commutativity of the left-hand triangle tells us that the first coordinate of the element (a, b) of $A \times B$ is precisely the element a of A. Analogously, the commutativity of the right-hand triangle tells us that the second coordinate of (a, b) is the element b of B.

So, and summing up, any pair of elements $a: 1 \to A$ and $b: 1 \to B$ determine a *unique* element $(a, b): 1 \to A \times B$ of $A \times B$. Thus I think this provides us with enough justification for calling the map (a, b) *the ordered pair* determined by the elements $a: 1 \to A$ and $b: 1 \to B$. But there is more. This way of construing the concept of *binary products* and of *ordered pairs* also satisfies ROP from Part II, as we will now show.

Let then A and B be sets in CS, and let $a: 1 \longrightarrow A$, $a': 1 \longrightarrow A$, $b: 1 \longrightarrow B$ and $b': 1 \longrightarrow B$ be arbitrary elements. Thus, in addition to the commutative Diagram 16 above, the following diagram also commutes

Diagram 17



According to ROP

$$(a, b) = (a', b') \text{ if and only if } a = a' \text{ and } b = b'$$

So let us suppose first that $(a, b) = (a', b')$. Hence, due to the commutativity of Diagrams 16 and 17, all the following equalities hold

$$a = p_A\,(a, b) = p_A\,(a', b') = a' \quad \text{and} \quad b = p_B\,(a, b) = p_B\,(a', b') = b'.$$

And conversely, if $a = a'$ and $b = b'$ then, from the commutativity of Diagram 17, all the following equalities hold

$$a = a' = p_A\,(a', b') \quad \text{and} \quad b = b' = p_B\,(a', b')$$

Therefore, both the map $(a', b')\colon 1 \to A{\times}B$, and the map $(a, b)\colon 1 \to A{\times}B$ make Diagram 16 commute. By the uniqueness of this latter map, we conclude that

$$(a, b) = (a', b')$$

Thus ROP is indeed satisfied in CS[19].

As we explained earlier in this section, definitions by universal properties—such as those of sets like 1 and $A{\times}B$—emphasize, *not* what the elements of the sets are, but how these sets are related to others with a similar "structure". And it is precisely this what guarantees that ROP is indeed satisfied for the elements of cartesian products.

Concluding Remarks

The formulation or analysis within ZF of the concept of *natural number* presents a similar problem to the one discussed here concerning the definition of the elements of cartesian products[20]. In 1965 the American philosopher Paul Benacerraf

[19]It is instructive to compare this proof with, for example, the one corresponding to Kuratowski's definition of *ordered pair*. See, *e.g.* the proof given in Enderton HB (1977), p.36.
[20]In this case too, CS offers a solution very much in the spirit of the one presented in this paper,

argued that natural numbers could simply not be sets (in ZF) because we cannot tell which sets they exactly are[21]. The same happens with binary products in ZF, and in both cases the "culprit" is, in my view, the axiom of extensionality, given the way I proposed to interpret this axiom. It is important to notice here that the basic concept in ZF—*membership*—is playing a crucial role in this axiom: if ZF cannot tell us exactly, for some given set *S*, which elements belong to it, ZF has failed to determine the set *S* itself. In this paper this set was "the" cartesian product $A \times B$ of any two given sets *A* and *B*.

In CS the approach is, as we have seen, completely different, starting from the fact *map* is now a basic concept and *membership* a derived one. Moreover, uniqueness of sets such as cartesian products, is no longer a demand of the theory. In addition, and as we saw in Part IV, CS shows that what one can *do* with one product, one can *do* it with absolutely *any* other product, for they are all isomorphic. And the same happens with other set-theoretic concepts such as one-element sets, natural numbers[22], empty *sets*, power sets, etc. What does matter in CS is the uniqueness of *maps*, and this is always guaranteed in all constructions or definitions by universal properties. And of course, this also applies in the case of *elements*, for these are (re)defined as maps with domain 1, which in turn is a set characterized by a universal property and hence unique up to isomorphism.

We saw then how for any given sets *A* and *B* in CS, *any given elements a* of *A* and *b* of *B* do indeed determine *a unique element* of $A{\times}B$, but now this element is a map. If we were to consider some other product for *A* and *B*, say *C*, but the same elements *a* of *A* and *b* of *B*, then these elements would also determine a unique element, but this time of *C*.

The characterization of products is one applicable in any category[23] and it can be easily extended to products of any number, finite or otherwise, of objects.

So let us consider an arbitrary family of objects $\{A_i\}_{i\in I}$ in some given category. Then *a* product for this family consists of an object, say *P*, together with a family of morphisms

$$P \xrightarrow{p_i} A_i$$

with $i \in I$, such that for any object *C* and any family of morphisms

$$f_i : C \longrightarrow A_i$$

also indexed by *I*, there exists a unique map

since "the" set of natural is also defined by a universal property. See, *e.g.* Awodey S (2006), p.217, although there he is not interested in contrasting it with the concept of *natural number* in ZF.

[21]Benacerraf P (1965), pp. 47-73. However, his discussion goes well beyond this point.

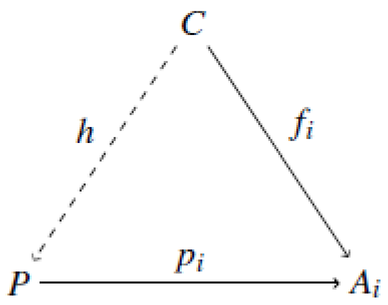
[22]For the natural numbers the dilemma arises when one tries to define "the" successor of any given natural number. This case has in addition quite interesting consequences dealing with the principle of mathematical induction and the simple recursion theorem.

[23]In fact, all characterizations by universal properties are applicable in any category.

$$h: C \dashrightarrow P$$

which makes the following diagram commute for each $i \in I$

Diagram 18



This more general and abstract setting makes it easy to see two somewhat extreme cases: one in which the index set I is just a singleton, and the other one in which it is empty. Let us then see the first case.

Consider an object A in some given category. Then a product for A must consist of a single morphism *to A* itself. We only need now to find the *domain* of such a morphism. The natural choice is, of course, the identity on A

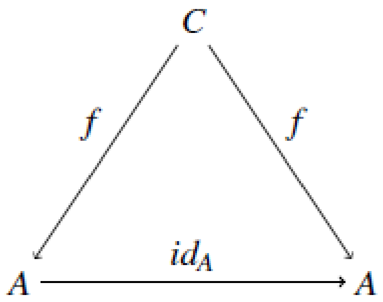
$$A \xrightarrow{id_A} A$$

since we do not know anything else about A. So let us see whether A together with its identity morphism is indeed a product for A. Let then C be an arbitrary object and $f: C \longrightarrow A$ any morphism from C to A. Notice that the relevant structure for this case looks like this

$$\nabla \longrightarrow A$$

So we have the morphisms $f: C \longrightarrow A$ and $id_A: A \longrightarrow A$, and we need to find a morphism, say $h: C \longrightarrow A$ such that $id_A h = f$. But due to the defining properties of identity morphisms, f is clearly such that $id_A f = f$. In other words, Diagram 19 below is commutative

Diagram 19

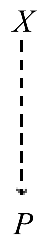


Moreover, and also due to the defining properties of identity morphisms, f: $C \longrightarrow A$ is the only morphism making Diagram 19 commute. So A together with its identity morphism is indeed a product for A.

Let us then consider our last case, *viz.* when the index set is empty. A product for zero objects must still consists of an object, say P, together with… no morphisms at all. But in order for P to be a product for zero objects it must satisfy a universal property. The relevant structure here is just P, so we must consider a single but arbitrary object of the category, say X.

The universal property then requires that there be a unique morphism

Diagram 20



such that… nothing, there are no further conditions that should be satisfied, since the index set is empty and hence there are no projection morphisms. So a product for zero objects is precisely a terminal object.

We have now come full circle. We started with the basic concept of *element* in ZF, and focused on one-element sets. We saw that for any set A and any one-element set 1, there is *a unique function* from A to 1. We also saw another important property any one-element set has: there is a bijection between, on the one hand, the elements of any set and, on the other, *functions* from any one-element set to the given set. These were the first steps of my proposal for making a conceptual change in which sets are to be thought of not in terms of their elements, but in terms of functions, either from them to other sets, or from other sets to them. All this set the stage for introducing the concept of *category* in which the basic notion is that of *morphism* and whose axioms state very general properties of the concept of *function* as this latter is used in the practice of mathematics.

Once we were in the framework of category theory, I introduced a few axioms for the category sets and maps that were necessary for the main topic of this paper. In Part II we saw the dilemma within ZF arising from the attempt to characterize cartesian products in terms of their elements. Then, in Part III I argued that CS offers us a solution to this dilemma, in my view entirely satisfactory. And *qua* solution, it involved in a crucial manner one-element sets (in the sense, of course, of CS).

I hope that some readers find the analysis presented here of the interplay between ZF and CS interesting in its own right. In the case of the definition of an *ordered pair* (*a*, *b*) within ZF, we saw that not only must one choose a specific set, among many other equally good candidates, for being "the" ordered pair (*a*, *b*), but that the grounds for choosing one candidate over all the others are *not* mathematical. Most textbooks (perhaps all of them) on Zermelo-Fraenkel set theory, simply bypass this issue and move on to "construct" cartesian products and to prove results about them. The content of this paper may be used as a pedagogical pathway for teaching the concept of *products* in category theory: start with a simple, non-mathematical problem within ZF—whose concepts are seemingly easier to understand than those from category theory—, then continue translating basic concepts from ZF into their counterparts in CS. In this way, the categorical concepts of *map*, *terminal object* and *binary products* get grounded in a comparatively more familiar terrain. And it is precisely in the categorical concept of *binary products* where the original problem dissolves completely.

There are other instances of this interplay between ZF and CS. For example, the apparent difference between how the axiom of choice is formulated in ZF and in CS. Showing that they are saying the same thing, leads to the construction of a category of sets different from those in CS, in which the axiom of choice fails[24]. The definition of "the" successor of a natural number in ZF presents a similar problem as the one discussed in this paper, since successors in ZF must, once again, be specific sets[25]. Moving to the, unique up to isomorphism, set of natural numbers in CS, not only solves the problem or ambiguity in ZF concerning the definition of successors, but it also serves to express the simple recursion theorem by universal properties. The proof of this theorem in ZF is convoluted and quite involved, and in my view it obscures the reason why it works for "the" set of natural numbers. One may even argue that the so-called construction of the natural numbers in ZF was done in an *ad hoc* manner in order for the principle of induction to follow directly from it. In any case, the principle of induction is also a theorem in CS and it is quite easy to prove. There is yet another instance of the interplay between ZF and CS: Zermelo's axiom of separation. Zermelo's contemporaries were not satisfied with its formulation, and Zermelo tried to reformulate it in various ways. The accepted solution was to bring *first order formulas* into the theory, with the result that the axiom became an axiom *schema*, one axiom for each well-formed formula. The original version stated that for each well-defined *property* of sets, one can always separate from a given set *A*, those and only those elements of *A having that property*, which results in a subset of *A*. The current version states that with each first-order well-formed formula, there is a set whose elements are those and only those elements

[24]See Pallares-Vega I (2020).
[25]See Pallares-Vega I (2024).

of *A satisfying that formula*. The problem Zermelo faced consisted in that he was never able to give a precise meaning to these well-defined properties. In any case, the move from *properties of sets* to *sets satisfying a well-formed formula*, seems to be at odds with the spirit of the other axioms Zermelo initially proposed for the concept of *set*[26]. In CS the notion *property* is quite simple and the truth-value axiom captures what I think Zermelo wanted these well-defined properties *to do*: to divide a given set *A* into two mutually exclusive parts, one with all those elements of *A* satisfying the property, and the other part with all those elements not satisfying the property. The truth-value *set* is defined by a universal property and in a more general context, truth-value objects in a category are called *subobject classifiers* and, along with other axioms, give the category in question the structure of what is called a *topos*. The concept of *subobject classifier* is certainly quite abstract and difficult to understand to non-specialists. Thus Zermelo´s well-defined properties can also be used as a pathway into the concept of *subobject classifier*[27].

All of these are cases in which one starts with different issues raised by certain concepts, axioms or theorems within ZF, and then one moves to CS in order to see the different ways in which they are formulated there. I hope to have shown how, by moving from ZF to CS, one may understand better the meaning of certain general and abstract concepts, axioms and theorems from category theory[28].

In Leinster T (2014), the author makes two somewhat strong claims: According to one of them ZF is, in sharp contrast with CS, practically of no use to professional, working mathematicians. He further claims, emphasizing the verb, that CS *is* set theory. Even if the first claim is right, my reply to it is why should mathematical theories be restricted to professional, working mathematicians? Why not try to make them accessible to anyone interested in them, at least at a basic level?[29] Or, why abandon a mathematical theory only because it is of no use to working mathematicians? The ideas I have presented in this paper, were intended to show how one can use ZF to *understand* quite abstract *concepts* in CS and, more generally, concepts in category theory. I think ZF is a wonderful theory in its own right and I still find it remarkable that many mathematical concepts can be recast within it. None of the theories mentioned by Leinster T (2014), apart from CS, shares this feature with ZF. And I disagree entirely with the second claim, *viz.* that CS *is* set theory. Like many other areas of human activity, mathematical practice (and *perforce* mathematics itself), changes over time and it is constrained by many factors (historical, cultural, perhaps even political). And I believe that this is a good thing, for there may come along in the future other, more interesting and challenging theories about the ubiquitous and seemingly modest concept of *set*.

[26]Zermelo F (1908a).
[27]See Pallares-Vega I (2022) for a detailed account of this approach.
[28]I find it noteworthy that in the construction or definition of sets in which the quantifiers and the logical connectives play a fundamental role, dilemmas like the one discussed here, do not arise within ZF. These are the cases of, for example, unions and intersections.
[29]Cheng E (2023) is an excellent example of how to make category theory accessible to outsiders.

Acknowledgements

I thank Professor Laura Campos-Millán, an expert in the philosophy of language and an esteemed colleague at the Autonomous University of the State of Morelos. She kindly explained to me the phenomenon I have discussed in this paper and which I, perhaps wrongly, have labeled *dilemma*. Any mistakes in the explanation I give here of the linguistic phenomenon called *undetermination of the referent* are mine.

I also wish to express my thanks to Luis Armando Cano-Armas, currently a PhD student in applied mathematics at the National Autonomous University of Mexico, for having drawn all the Diagrams in LaTeX.

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