Cognitive and Metacognitive Skills on Number Theory in the Proving Process of Math Teachers

Mathematics teachers’ proving process is important for the development of their proof skills. Developing teachers’ proving skills can contribute to their students’ meaningful learning of mathematics. For example, teachers showing simple proofs about number theory make it easy for students to understand the concepts of multipliers and factors and the concepts of the greatest common divisor and the least common multiple of their students. For this reason, this study was conducted in order to examine the cognitive and metacognitive skills performed in the proving process by math teachers. In the study, we used the model of case study from qualitative research design. Six elementary and eight secondary math teachers participated in the study. Data in the study were collected through task based interviews (think-aloud protocol), documents and observation forms. The collected data were analyzed with content analysis method. The results of the study show that, based on our operational definition of cognition and metacognitive skills, elementary math teachers used generally used cognitive skills, while secondary mathematics teachers performed metacognitive skills.

Keywords: proof; math teacher; number theory; elementary; secondary

Introduction

Literature emphasizes that proof is important for mathematics education. Because proof requires high level mathematical thinking and work systematically. However, many students think that proof is difficult to write and understand (Öztürk & Kaplan, 2019). Thus they are not volunteer for proving (Knuth, 2002; Raman, 2003). Since students’ knowledge and thoughts about proof are related to their teachers' knowledge and thoughts about proof, it is important to examine the knowledge and thoughts of mathematics teachers about proof. Math teachers with advanced proving skills can work systematically and perform high level mathematical thinking skills. This may affect their students’ math learning process. Because proving requires advanced cognitive skills and awareness. Awareness of cognition is defined generally as metacognitive skills (Öztürk & Kaplan, 2019). Making proof are quite important for math teachers, but the math curriculum in Turkey does not have enough proof related subjects. International examinations for elementary and secondary school students such as TIMSS and PISA report that Turkish students have low mathematics skills. The negative results of both TIMSS and PISA led us to conduct this study. This study aimed to compare cognitive and metacognitive skills of elementary and secondary math teachers in the proving process.

Proof in Mathematics Education and Mathematical Proof

Mathematical proof is to determine whether a claim is correct or not by
using mathematical symbols and formulate. The proof in mathematics education is to convince the students of the accuracy or inaccuracy of mathematical propositions whose accuracy has been demonstrated (Aydoğdu-İskenderoğlu, 2016, p. 66). These definitions show that there are small but sharp differences in concept of proof in math and math education. The main point of distinction is that there are a few proofs in mathematics education while the proof in mathematics is single. Therefore, the purpose of the proof is different in these two disciplines. In mathematics, the first demonstration of the theorem's accuracy is proof, and all subsequent representations of accuracy (demonstration of accuracy in different ways) are representations. The proof contributes to the systematic development of math, and once a theorem is proved, the proof of other theorems that will use it as a lemma begins. Unlike math, in math education, proof is used for understanding theorems, but also for developing mathematical thinking and understanding mathematical concepts (Dawkins & Weber, 2017).

Many mathematics educators consider proof as an important part of math courses and point out that it is necessary to know how to make proofs for mathematical applications (Öztürk, Akkan & Kaplan, 2019). Hanna and de Villiers (2008) stated that in order to gain skills to make proofs, individuals should meet proofs from early ages. These researchers also stated that math curriculum and teachers’ attitudes and knowledge are important to make proofs. Generally, Number Theory Course includes the most frequently proof in math curriculum. This course and proof is related to the subjects in K-12 education. Therefore, it is important to consider the proofs in the number theory course. Besides, number theory is one of the basic courses. Number theory forms the basis for other courses such as algebra. If math teachers gain high level of skills to make proof, they can teach math as conceptional and justify accuracy of a claim. Acceptance of a claim is related to validity of proof. For a proof to be valid, it must be both reliable and true of each claim or premise (Tall & Mejia-Ramos, 2010). All this is possible through systematic study and the development of metacognitive skills.

Cognition and Metacognition

Cognition is the structure that involves the all operations in the process of completion of a task (Öztürk & Kaplan, 2019). It is a mental condition that can be used by individuals (Husamah, 2015), but it does not require an advance level of skill. In other words, it refers to the processes and strategies used by the reader to complete a task (Öztürk & Kaplan, 2019). For example, cognition is seen as a recollection of an individual’s knowledge by Forrest-Pressley and Waller (1984). A learner’ being aware of his/her tactic and strategy knowledge or monitoring his/her cognition leads us to the concept of metacognition (Winne & Azevedo, 2014).

Metacognition means the person being aware of their own knowledge and organizing their own knowledge (Flavell, 1976). In another definition, metacognition was expressed as someone being aware of their own thinking procedures and being able to change and organize his own thinking (Spruce & Bol, 2015). There are two basic categorizations of the concept of
metacognition. The first of them is the knowledge of cognition -the person being aware of how they learn-, and the second is regulation of cognition -planning, control, monitoring and evaluate- (Garner & Alexander, 1989).

The distinction between cognition and metacognition is related to how knowledge is used and what is the object of the process. The skills required to complete a task (such as knowing strategies, using representations) are cognitive; the awareness of these skills and thinking on these skills are metacognitive skills (Öztürk & Kaplan, 2019; Winne & Azevedo, 2014). Weinert (1987) explained the metacognition as awareness of cognition and he said that metacognition is defined as second degree of cognition, in other words, as thinking about thinking. Winne and Azevedo (2014) described metacognition as cognitive features of the information on which a learner operation. When evaluating a subject or performing a task, note-taking, recognition of operational errors, and comparison with the result was evaluated as metacognitive. When performing the same task, instead of noting a note, copying and using a ready-to-use formula that is memorized are evaluated as cognitive. These skills provide a way to distinguish between cognitive and metacognitive skills (Öztürk & Kaplan, 2019). This distinction can be explained as follows; while metacognitive skills require awareness, high-level thinking, or critical thinking, cognitive skills are used to perform the task without the use of any similar skills (automated skills) (Akın, 2013, p. 123).

The Purpose of the Study

Literature in mathematics education shows that there has been an increase in studies on mathematical proof in recent years. This demonstrates the importance of proof. When the developmental stages of the studies in mathematics education are examined, it is seen that first descriptive researches and then studies examining opinions were conducted. In the third stage, there are studies investigating cognitive structures. As a matter of fact, the literature about proof in math education shown that descriptive researches and studies on opinion analysis were inadequate, but studies on cognitive structure was inadequate. Öztürk and Kaplan (2019) collected the cognitive structures of secondary math teachers in the proving process in two themes as cognitive and metacognitive. However, the study was limited to secondary math teachers and did not reveal the situation among elementary math teachers. Uncovering the proving process of primary and secondary mathematics teachers in Turkey, while indicating the current status of mathematics teachers will make it possible to compare the primary and secondary school mathematics teachers. In this context the present study aimed to determine the cognitive and metacognitive skills performed by math teachers in the proving process. In accordance with this purpose, we searched for answers to the following research questions:

1. What are the cognitive and metacognitive skills performed by elementary math teachers in the proving process?
2. What are the cognitive and metacognitive skills performed by secondary math teachers in the proving process?
3. What are the differences between cognitive and metacognitive skills of elementary and secondary math teachers?

**Method**

**Research Model**

We used the qualitative case study methodology in the study. The case study aims at an in-depth examination and explanation of a phenomenon or a case using variety data sources (Creswell, 2007). There are some categorizations as explanatory, descriptive and multiple-case study of the case study (Baxter & Jack, 2008). In this study, we selected multiple-case study. Multiple-case study can use to explore differences within and between cases (Baxter & Jack, 2008). Multiple case studies are preferred by researchers in order to make comparison between cases. In this study, we used the multiple-case study because we aimed to compare cognitive and metacognitive skills in the proving process of elementary and secondary math teachers.

**Participants**

The study was finally conducted with 14 in-service math teachers. We first interviewed 28 elementary and 22 secondary math teachers for the selection of the sample. These 50 teachers where asked general questions like “Do you think that you are qualified in proving?”, “Can you perform a proof without assistance from others?” Through this unstructured interview, many teachers who considered themselves inadequate since they expressed that they could only perform memorized proofs were removed from the study. Consequently, 14 math teachers in total, 6 elementary math teachers and 8 secondary math teachers participated in the study voluntarily. Other teachers stated that their thinking level was not adequate for making proofs. Fifty percent of elementary and secondary math teachers were female, and fifty percent of them were male. Two of the elementary math teachers and three of the secondary math teachers had 1-5 years of experience, three of the elementary and secondary math teachers had 6-10 years of experience, one of the elementary and secondary math teachers had 11-15 years of experience and one of the secondary math teachers had 16-20 years experience. In this study, all participants informed regarding the study and ensured that their personal information would not be shared with anybody. The ethnic groupings were Turkish. Code names were used for participants in direct transfers from participants. The creation of code names has been made possible by $T_{ij}$ matrix encoding. In this coding Teacher symbol represents the Researcher’s code.
Metacognitive skills can generally be examined through verbal reactions given by the person to the situation they encounter in the proving process. For example, in a problem faced during the problem solving process, one’s own difficulty in solving and needing help may reveal metacognitive skills with the help of verbal expressions (Veenman, Hout-Wolters, & Afflerbach, 2006). However, determining metacognitive skills through verbal expressions may not be sufficient in every case. For example, the metacognitive planning skill of determining the suitable strategy in the problem solving process and making systematic operations accordingly requires examination of not only verbal expressions, but also written expressions (Veenman et al., 2006). In this context, in order to demonstrate the operations in the proving process and also determine metacognitive skills from verbal expressions, task based interviews were used in the study. Observation forms were utilized to determine non-verbal section of skills performed in more detail. Garner and Alexander (1989) emphasized that since determining metacognitive skills is difficult, using multiple data collection tools can be beneficial.

In the first stage, four written questions on proving were prepared to be used in the task based interviews. The prepared questions were asked to two faculty members in the math education field and to two in the mathematics field and their opinion were taken. In line with their opinion, two questions for the task based interview were prepared based on symbolic and verbal expressions. The chosen questions were implemented to one elementary and one secondary math teacher. As a result of the implementation, it was determined that the problems were solvable for teachers, and that they could be sufficient in terms of demonstrating the process. In task based interviews, the teachers were asked to make the proof by thinking aloud protocol. By making those thinking aloud protocol, teachers’ immediate thoughts and question they ask themselves in this process were attempted to be revealed. The task based interview questions used in the study are presented in Table 1.

Table 1. The task based interview questions they asked themselves used in the study and the reasons for their use
Question content

Based on Verbal Expression
Show that the expression “On the integers set, every number divisible by 3 and 4 can be divided into 12” is correct.

1. Is this proposition correct? Why? (How they do intuitive reasoning will be examined).
2. Is the proof you carried out valid? Why and how did you decide?
3. Why did you do the operations you did? (It is asked suitably for every performed operation).
4. Show that the expression “On the set of integers, every number divisible by 7 and 9 can be divided into 63” is correct.

Based on Symbolic Expression
Show that $p(n); 3|\left(2^{2n} - 1\right)$ is true for $\forall n \in \mathbb{Z^+}$.

1. What does the proposition tell you? (Whether symbolic expressions are understood will be examined.)
2. Is this proposition correct? Why? (How they do intuitive reasoning will be examined).
3. Is the proof you carried out valid? Why and how did you decide?
4. Why did you do the operations you did? (It is asked suitably for every performed operation).

Selection reasons of first question are following; this question is at a suitable level for all teachers to solve, it accuracy can be demonstrated through different proving methods, it is suitable in terms of examining whether all the conditions are checked for valid proof and Because it is easy to express verbally, it allows examining whether it uses symbolic expression or not.
Selection reasons of second question are following; it contains symbolic expressions, it is suitable in terms of examining whether symbolic expressions are understood and it is important in terms of determining whether all conditions are checked in the generalization of the proof.

For the observations, a semi-structured observation form developed by the researchers was used. The form was revised and reorganized after the interviews and prepared in the 3-Likert type, in accordance with the task based interview. If the determined skill was not performed in any problem, the option “0”, if it is performed in one problem, “1” and if it was performed in two problems, “2” was ticked. Moreover, records were made in the explanation section for various skills exhibited by the teacher within the process.

Data Analysis

The data were analyzed with content analysis method. In content analysis, the collected data were first encoded. In other words, we constructed subcategories. During initial encodings, only the skills were determined. Afterwards, encoding and questions asked were re-examined, and replies from the teachers and studies in the literature (Cozza & Oreshkina, 2013; Schoenfeld, 1985; Yang, 2012) were utilized for demonstrating whether the skill is cognitive or metacognitive. In content analysis, the data were encoded by the first researcher and a selection of suitable-not suitable type was made by
the second researcher. At the end of the selection, consistence between the two researchers was checked with the Cohen Kappa formula. Consistence between the researchers was calculated as .84, which indicates that value coding is highly reliable (Landis & Koch, 1977). We collected sub-categories on categories accordingly common specialization. In the study, task based interviews took about 30 minutes on average. In cases where whether the cognitive or metacognitive characteristic of the skill couldn’t be decided from the replies, the teacher was interviewed again. Direct quotations from the teachers’ opinions and solutions were added.

Validity and Reliability

Internal validity and external validity studies were conducted to confirm the validity of the study. The internal validity of the study was increased by using different data collection tools together. The external validity of the study was increased by explaining the participating in detail and making direct excerpts from the statements of the participants.

To ensure the reliability of the study, we have ensured that the research process is consistent within itself (research model, participants, data collection tools and analysis of collected data). The research process was carried out by the first researcher while the second and third researchers controlled the research process continuously.

Results

Results of the study are presented in three titles. Firstly cognitive and metacognitive skills performed in proving process of elementary math teachers, second cognitive and metacognitive skills performed in the proving process of secondary math teachers, and the comparison of cognitive and metacognitive skills in proving process of elementary and secondary math teachers.

Cognitive and Metacognitive Skills Performed in Proving Process of Elementary Math Teachers

Cognitive and metacognitive skills performed in proving process of elementary math teachers collected in three categories. These categories are verification, explanation and generalization.

Cognitive and metacognitive skills performed accordingly verification by elementary math teachers

We determined that the elementary math teachers participating in the study performed three cognitive and three metacognitive skills accordingly verification. The cognitive skills were “Reading proposition for the first time”, “Reading the proposition step by step” and “Proposition controlled as heuristic”. The metacognitive skills were “Read the proposition repeatedly”, “Making a guess” and “Draws a diagram or a table”.
The elementary math teachers were reading the proposition which includes two verbal statements step by step. Observation form showed that five elementary math teachers \( T_{E1,E2,E3,E4,E5,E6} \) read proposition for the first time. In other words they read proposition as whole. While the form showed that \( T_{E2} \) read proposition step by step. The same teacher intuitively checked the accuracy of the proposition. This sub-category was detected with observation form and task based interview. \( T_{E1} \) said that “The proof I made was not valid, because the proof I made was very simple and ordinary. In other words, I did not use any symbols. I only wrote the explanation.” This expression stated that he didn’t aware that whether his proof is valid or not. Thus we considered that sub-category of “proposition controlled as heuristic” was cognitive skill.

Two of the elementary math teachers read the proposition repeatedly. Opinion of teacher showed that this sub-category was metacognitive skill. For example, \( T_{E2} \) said that:

“I read it a few times to find out what should I do. In the first part of the proposition, it says that every integer is divisible by 3 and 4. Does it say every because there are special cases that are not?”

This reply of the teacher showed that he was aware of why he is reading the proposition repeatedly. Thus we decided that this sub-category was a metacognitive skill. One of the participant \( T_{E2} \) said that he guessed proof of proposition. And then, he controlled in his mind. This sub-category is necessary metacognitive controlled, thus we detected it as metacognitive skill. Participates of the study \( (T_{E3,E4,E5,E6}) \) drew diagram or table and their explanation showed that they aimed comprehension of proposition with drawing diagram or table. Thus we decided that sub-category of draws a diagram or a table was a metacognitive skill.

**Cognitive and metacognitive skills performed accordingly explanation by elementary math teachers**

We determined that the elementary math teachers participating in the study performed two cognitive and five metacognitive skills accordingly explanation. The cognitive skills were “trial-error strategy” and “Proves the opposite of proposition”. The metacognitive skills were “Determined key idea”, “Proved in his/her mind before written proof”, “Making a decision”, “Questions in proving process” and “Establishes relationship between steps”.

While many teachers decided whether they can make the proof or not, some teachers \( (T_{E5,E6}) \) made this decision without thinking and prefer to see the proof by trying. The teachers try to proof of proposition, and then they
make decision which is not making proof. In this sense, trial-error strategy for proving process was considered as cognitive skill. A teacher $T_{E2}$ proved opposite of proposition. This proof was not valid and the teacher was not aware of it. Thus sub-category of “Proves the opposite of proposition” was considered as cognitive skill.

We detected that a teacher $T_{(E1)}$ determined key idea for proof. Expression of teacher showed that he was aware of the importance of determined key idea. For this reason he determined a key idea. Thus we considered that sub-category of “Determined key idea” was metacognitive skill. Four of elementary math teachers ($T_{E2,E3,E5,E6}$) proved in his/her mind before they write to proof. This indicates that they check the proof in their minds, thus we detected it as metacognitive skill. A teacher $T_{E1}$ stated that he didn’t show proof of proposition because his knowledge level was not adequate. Another teacher $T_{E2}$ said that “I think the proof I made is not valid, because I didn’t use any symbolic expression.” This expression showed that teacher evaluated the proof himself. Thus we thought that sub-category of “Making a decision” was metacognitive skill. Three teachers ($T_{E1,E2,E3}$) asked themselves questions in proving process. Because of questioning is a part of metacognition, we considered that this sub-category determined with observation form was metacognitive skill. In the observation form of $T_{E1}$, we determined that the participant correlates with the proof steps. Relationship is necessary metacognitive monitoring. Thus we considered this sub-category as metacognitive skill.

Cognitive and metacognitive skills performed accordingly generalization by elementary math teachers

We determined that elementary math teachers performed three cognitive skills and one metacognitive skill accordingly generalization. The cognitive skills were “Justifications dependent on authority”, “Justification based on example” and “Pattern generalization”. The metacognitive skill was “Symbolic justification”.

A participant $T_{E2}$ justified dependent on authority making proof of proposition. $T_{E2}$ said that:

“... when all multiples of 4 are divided by 3, they give the remainder of 1. So if we consider the opposite, i.e. subtracting 1 from the forces of 4, the number is divided by 3. I’ve seen this proposition is proven this way before.”
These expressions give the impression that the teacher bases his proof justification to a source he saw before and remained ritual to the authority. We determined that four of the elementary math teachers \((T_{E3,E4,E5,E6})\) made justification based on example. For example,

“Let’s try a few numbers divisible by 3. Then let's try a few numbers divisible by 4. Each number that can be divided into 12 can be divided by… [Firstly teacher created a table] … Let us create a table for hundred. Let us include numbers which can be divided both into 3 and 4 in this table… [Teacher showed on the table.] …” \([T_{E6}]\)

\(T_{E6}\)’s these sentences stated that the teacher was making example based justification. Task card of the teacher point out this skill is performed. Fig. 1 is an example for justification based on example. Explanation of the teacher is presented with figure.

[03.04] R: Is your diagram enough to prove proposition?  
[03.19] \(T_{E6}\): No, not from the point of the proof methods we know. In other words, it does match proof methods like reasoning and deduction. However, I proved it using a model.  
[03.56] R: Are the proofs made with diagram or model valid?  
[04.05] \(T_{E6}\): The proofs made with diagram or model are valid. Because it gives the correct result.  
[04.23] R: Then how can we show that the number which can be divided by 7 and 9 can also be divided by 63?  
[04.47] \(T_{E6}\): We can do it the same way. This time a table for thousand must be formed.

This type of justification requires the teacher only to use knowledge. He does not need to think about cognition. Thus this sub-category was considered as cognitive skill. We detected that five of participant \((T_{E2,E3,E4,E5,E6})\) made generalization pattern, while showing the accuracy of the proposition instead of inductive method. Below sentences of \(T_{E2}\)’s provides an example for generalization pattern.
“I will divide each value I find into 12. This is 12 once, this is 2 twice, and this 3 times … if it goes on like this, then we can reach a result like 12\cdot n. In other words, we can reach the result by generalizing the pattern.”
These expressions indicated that the teacher made proof via generalization pattern.

This skill is only necessary for cognitive operations and does not require any metacognitive awareness. Thus we considered that the sub-category of “Generalization pattern” was cognitive skill.

Two of the elementary math teachers \((T_{E2,E4})\) made symbolic justification for proof. These teachers used symbolic expressions task card for proof and they emphasized that proof is valid, because they made proof as systematically and used symbolic expression. They used metacognitive monitoring and metacognitive awareness to determine the validity of the proof. Thus sub-category of “Symbolic justification” was considered as metacognitive skill.

Cognitive and Metacognitive Skills Performed in Proving Process of Secondary Math Teachers

Cognitive and metacognitive skills performed in proving process of secondary math teachers collected in three categories. These categories were verification, explanation and generalization.

Cognitive and metacognitive skills performed accordingly verification by secondary math teachers

We determined that secondary math teachers performed three cognitive and four metacognitive skills at accordingly verification. The cognitive skills were “Reading proposition for the first time”, “Reading the proposition step by step” and “Basing on the previous theorem”. The metacognitive skills were “Read the proposition repeatedly”, “Making a guess”, “Draws a diagram or a table”, and “Questioning for corrected proposition”.

One of the secondary math teachers \((T_{S5})\) read step by step the proposition. When she read the proposition first time, she read it as partially. However other teachers read proposition as a complete or whole for the first time. Skills with reading of secondary math teachers were detected via observation form. Three of the teachers verify the proposition by based on the previous theorem. \(T_{S5}'\)s sentences “Since there are relatively prime, proposition is correct. I heard that there is such a rule for the number containing relatively prime numbers, in a book.” and again the same teacher said that “What is this theorem? I tried to remember it… I am thinking whether I proved it or not.” These expressions showed that the teacher made decision to accuracy of proof via rote.
The observation form showed that five secondary math teachers read the proposition repeatedly. Teachers showed that they did not understand what they read as the reason for re-reading the proposition of proof. This state showed that they are aware of their understanding, in other words they performed metacognitive awareness. Thus we considered that sub-category of “Read the proposition repeatedly” was metacognitive skill. Five of secondary math teachers \( T_{51,54,55,56,58} \) made guess toward proof of proposition. For example, \( T_{57} \) said that “I think it can be solved from the difference with two squares. Perhaps I may get a result from it. But I won’t be able to solve it.” \( T_{57} \)’s expression indicated that teacher was aware of their own knowledge. In other words she has cognitive knowledge. Thus this sub-category was considered as metacognitive skill. One of the secondary math teachers \( T_{56} \) drew diagram for proof of proposition. \( T_{56} \) said that:

“Proof cannot be done with model. However, in order to see the proposition more clearly, drawing a diagram and creating a table is important. Of course it is not one of the proof methods, but makes the event more understandable in our mind.”

Expression of the teacher showed that she drew diagram for comprehension but she was aware that drawing a diagram is not proof. This situation requires metacognitive awareness. Thus sub-category of “Draws a diagram or a table” was considered as metacognitive skill. One of the secondary math teachers \( T_{57} \) questioning for corrected proposition. \( T_{57} \) said that:

“Is the result of 1 less double exponent of two, times three?... [He is asking himself questions by thinking out loud. He is proving by using different strategies mentally] ... I checked that it is correct with a few examples. Yes, it is. Proposition is correct but I could not reach a general result…”

The expression of the teacher indicated that she was questioning for corrected proposition. Questioning is an important part of metacognitive monitoring. Thus we considered that the sub-category of “Questioning for corrected proposition” was metacognitive skill.

**Cognitive and metacognitive skills performed accordingly explanation by secondary math teachers**

We determined that the secondary math teachers participating in the study performed two cognitive and three metacognitive skills accordingly explanation. The cognitive skills were “Trial-error strategy” and “Proves the opposite of proposition”. The metacognitive skills were “Determined key
idea”, “Proved in his/her mind before written proof” and “Justified accordingly axiomatic proof scheme”.

Four secondary math teachers \( T_{52,54,55,58} \) decided whether they can make the proof or not, some teachers made this decision without thinking and prefer to see the proof by trial-error strategy. For example, \( T_{54} \) said that “If nothing in my mind regarding solution appears, then I try to prove by trial-error.” This expression indicated that the teacher used trial-error strategy. In addition, the expression showed that he was not aware of proof. Thus we considered this sub-category was cognitive skill. Two of the secondary math teachers \( T_{53,58} \) showed the opposite of the proposition was accurate. \( T_{52} \) showed the opposite of the proposition. Researcher asked him “Would showing the opposite of the proposition to be correct be adequate in showing the accuracy of the proposition?” He answered the questions as “Yes. It is adequate in showing the opposite being correct.” \( T_{58} \) after proving the opposite of the proposition, said:

“I do not know if proof can be done from the opposite, because I had to reach twelve by starting with three and four. Let us try this, \( a=3k \) and \( a=4l \).” These expressions point out that this teacher has cognitive skill in the proving process. Because she has got wrong information and she was not aware that her information is false.

Six secondary math teachers \( T_{52,53,54,56,57,58} \) determined the key idea for proof and outlined the limits of the proof in general sense. For example \( T_{54} \) said that; “First of all I select the divisor numbers which are divisible into three and four. When I take any a number, it will be the divisible of 3 and 4.” These expressions show that the teacher determined key idea, and this key idea ensured detected limit of proof. In other words he aimed ease the proof. Therefore, since key idea required an advanced level of skill, we considered as metacognitive skill. Four the secondary math teachers \( T_{51,54,57,58} \) proved in his/her mind before written proof. For example, \( T_{51} \) said that: “When I read the question certain things appear in my mind. I check the accuracy in my mind.” It showed that the teacher solved in his mind before his written solution. This situation require controlled metacognitive. Thus we considered that sub-category of “Proved in his/her mind before written proof” was metacognitive skill. Six secondary math teachers \( T_{52,54,55,56,57,58} \) justified their proofs accordingly axiomatic proof scheme. For examples, we presented two teachers’ words under the texts.

“I believe that the proof I made is correct, because induction method shows accuracy for all numbers. In other words, if we accept that they
resembled dominos, when we trip one domino they all fall down. Now operation I make here provides for 1 and also for 2. Induction principle is always used for positive integers. Here it verifies for all integers” \([T_{S2}]\)

“I am sure that the answer is correct, because I considered all situations by examining step by step. In other words, I checked all situations. This strengthened my thought that the proof is valid. But I cannot say it is final. But I should have checked it; however, since I saw that I reached the result I should have reached as the result of the proof, I did not check it.” \([T_{S5}]\)

These expressions showed that teachers justified their proofs accordingly axiomatic proof scheme. This proof scheme requires high-order level of thinking and questioning all situations. Thus we considered that the sub-category of “justified accordingly axiomatic proof scheme” was metacognitive skill.

**Cognitive and metacognitive skills performed accordingly generalization by secondary math teachers**

We determined that secondary math teachers performed one cognitive skill and one metacognitive skill at accordingly generalization. The cognitive skill was “Justifications dependent on authority”. The metacognitive skill was “Symbolic justification”.

Four secondary math teachers \((T_{S2,55,56,SB})\) justified dependent on authority while evaluated of proof. For example, \(T_{SB}\) said that “I had seen this proof on the textbook. Therefore, I know that proof is correct.” These expressions indicated the teacher has evaluated accuracy of her proof accordingly textbook. In other words we can say that she evaluated based on authority accuracy of her proof. Justification which depends on authority doesn't require any metacognitive skill. Thus we considered that the sub-category of “Justifications dependent on authority” was cognitive skill.

Seven secondary math teachers \((T_{S1,53,54,55,56,57,SB})\) made justification based on symbolic expressions at the accordingly generalization. Fig. 2 and \(T_{SB}\)’s expressions are examples for symbolic justification.
Symbolic justification requires using of unknown and variable concepts, and high-order level thinking. Thus we considered that the sub-category of “Symbolic justification” was metacognitive skill.

The Compare of Cognitive and Metacognitive Skills in Proving Process of Elementary and Secondary Math Teachers

Table 2 shows the comparison of cognitive and metacognitive skills in proving process of elementary and secondary math teachers.
<table>
<thead>
<tr>
<th>Skills</th>
<th>Elementary math teachers</th>
<th>Secondary math teachers</th>
<th>Total</th>
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<tbody>
<tr>
<td><strong>Cognitive</strong></td>
<td></td>
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<tr>
<td>Verification</td>
<td>T_E1 T_E2 T_E3 T_E4 T_E5 T_E6 f T_S1 T_S2 T_S3 T_S4 T_S5 T_S6 T_S7 T_S8 f tf</td>
<td></td>
<td></td>
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<tr>
<td>Reading proposition for the first time</td>
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<td>X</td>
<td>X</td>
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<td>Reading the proposition step by step</td>
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<td>Proposition controlled as heuristic</td>
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<td>Basing on the previous theorem</td>
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<td><strong>Explanation</strong></td>
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<td>Trial-error strategy</td>
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<td>X</td>
<td>2</td>
</tr>
<tr>
<td>Proves the opposite of proposition</td>
<td>X</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td><strong>Generalization</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Justification dependent on authority</td>
<td>X</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Justification dependent on example</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Pattern generalization</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td><strong>Verification</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Read the proposition repeatedly</td>
<td>X</td>
<td>X</td>
<td>2</td>
</tr>
<tr>
<td>Making a guess</td>
<td>X</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Draws a diagram or a table</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td><strong>Explanation</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Determined key idea</td>
<td>X</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Proved in his/her mind before written</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Making a decision</td>
<td>X</td>
<td>X</td>
<td>2</td>
</tr>
<tr>
<td>Questions in proving process</td>
<td>X</td>
<td>X</td>
<td>3</td>
</tr>
<tr>
<td>Establishes relationship between steps</td>
<td>X</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Justified accordingly axiomatic proof</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td><strong>Generalization</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Symbolic justification</td>
<td>X</td>
<td>X</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 2 indicated the most frequent sub-category was “Reading proposition for the first time of math teachers”, and the least frequency sub-categories were “Questioning for corrected proposition” and “Establishes relationship between steps”. The most frequency sub-categories of elementary math teachers were “Reading proposition for the first time” and “Generalization pattern”. Both these sub-categories were cognitive skills. The most frequency sub-categories as metacognitive skill of elementary math teachers were “Draws a diagram or a table” and “Proved in his/her mind before written proof”. The most frequency sub-categories of elementary math teachers were “Reading proposition for the first time” and “Symbolic justification”. One of these sub-categories was cognitive skill, another was metacognitive skill. Elementary math teachers used generally cognitive skills, while secondary math teachers performed generally metacognitive skills.

**Conclusion and Discussion**

The present study aimed to determine and compare the cognitive and metacognitive skills performed by elementary and secondary math teachers at the proving process. Cognitive skills performed by math teachers with reading of proposition were “Reading proposition for the first time” and “Reading the proposition step by step”. Öztürk and Kaplan (2019) reported that skills with reading were cognitive skill. Yang (2012) expressed that reading step by step for the proving process is a cognitive skill. This result is consistent with their research. We detected that elementary math teachers controlled accuracy of proposition as heuristic. This result confirms earlier findings (Öztürk & Kaplan, 2019). Secondary math teachers in participating controlled accuracy of proposition based on the previous theorem. This finding is consistent with the findings of earlier studies (Öztürk & Kaplan, 2019). Many teachers decided whether they can make the proof or not, some teachers make this decision without thinking and prefer to see the proof by trying. In this sense, trial-error strategy for proving process is taken as cognitive skill. This finding is consistent with the findings of earlier studies on the proof and problem-solving (Cozza & Oreshkina, 2013; Öztürk & Kaplan, 2019). Many math teachers in participating prove the opposite of proposition. This result confirms earlier findings (Stavrou, 2014). We detected that elementary and secondary math teachers justified dependent on authority. This finding is consistent with the findings of earlier studies (Harel & Sowder, 1998; Öztürk & Kaplan, 2019). Elementary math teachers in participating justified dependent on example. This result confirms earlier findings (Harel & Sowder, 1998). Many elementary math teachers pattern generalization for proof. There are also research studies in this field declaring that pattern generalization is a cognitive skill (Čadež & Kolar, 2015; Öztürk & Kaplan, 2019).

We detected that reading the proposition repeatedly was metacognitive skill, because this skill requires to be aware of why reading proposition repeatedly. Yang (2012) reported that reading proposition repeatedly was metacognitive skill. This result is consistent with their research. Math teachers in participating made a guess. Making a guess was metacognitive skill, because
it is requires metacognitive controlled. Schraw (1998) pointed out that “making a guess by determining a target” is a metacognitive skill aimed at planning. Everson and Tobias (1998) also emphasized that guessing is a metacognitive skill. This finding is consistent with the findings of earlier studies. The results also revealed both elementary and secondary math teachers drew a diagram or a table. When the results obtained in this sense are compared, it can be stated that diagram or table use for comprehension of proposition. It is not really preferred in order to understand proof in secondary education level. Gourgey (1998) stated that drawing a shape is important for understanding, and this skill is a metacognitive skill. Depaepe, Corte and Verschaffel (2010) also express that drawing shapes or making a table skills during problem solving process are metacognitive skills. However, to the contrary of these, Yang (2012) stated that drawing a shape during proof operation to understand what is being read, is a cognitive skill. Yang’s (2012) work is a scale development work, and the fact that no interviews were made, may cause him assess skill as cognitive skill. Secondary math teachers was questioning for corrected proposition. We considered that this skill was metacognitive skill, because questioning is a part of metacognitive monitoring. Many research reported that questioning is a metacognitive skill (Öztürk & Kaplan, 2019; Öztürk et al.. 2019). Math teachers determined key idea for proof. Expression of math teachers showed that they are awareness of important of determined key idea. This finding is consistent with the findings of earlier studies (Öztürk & Kaplan, 2019; Raman, 2003). The results of the present study showed that math teachers proved in his/her mind before written proof. They stated that they made it for controlled. Thus we considered it as meacognitive skill. Öztürk and Kaplan (2009) emphasized that secondary math teachers made it in his/her mind first and then roll it in proving process. This result is consistent with their study. Many elementary math teachers made a decision with proof. Their expressions showed that they have got knowledge of cognition. Thus we considered that making a decision was a metacognitive skill. Öztürk and Kaplan (2019) reported that making a desision is a metacognitive skill. This result is consistent with prior research. Many elementary math teachers established relationship between steps of proof. We considered that this skill was a metacognitive skill, because relationship is necessary metacognitive monitoring. This finding is consistent with the findings of earlier studies (Schoenfeld, 1985; Yang, 2012). Secondary math teachers justified their proof accordingly axiomatic proof scheme. Axiomatic proof scheme requires high-order level thinking and questioning all situations (Tall & Mejia-Ramos, 2010). Participated math teachers justified their proof as symbolic. Schoenfeld (1985) stated that use of symbolic expression is a metacognitive skill. In contrast, Yang (2012) indicated that use of symbolic expression for reading proof is a cognitive skill.

This study was conducted with several limitations. The number of participants was the first of these limitations. Since the study was conducted according to the case study model, one of the qualitative research designs, it was aimed to make a detailed investigation and therefore the sample was limited. Future researchers may conduct studies examining the proof process by using mixed research methods with larger samples. Another limitation of
the study was that only the cognitive structure was considered. Recently, considering the developments in medical science, it is also possible to examine mental structures (with devices such as fMRI and PET). In this context, future researchers can mentally examine the process of making proof.

The results of this study showed that elementary math teachers used generally cognitive skills, while secondary math teachers performed generally metacognitive skills. Given that proof is important for school mathematics, teachers are expected to make better proofs and know what they are doing and why. In other words, the high number of metacognitive skills used by teachers is important for students to learn mathematics meaningfully. In this context, the use of metacognition-based instruction in the courses for proof-making skills may contribute to the development of metacognitive skills of teachers in the process of making proof.

References


