Mathematics Teachers’ Pedagogical Content Knowledge
Involving the Relationships between Perimeter and Area

The aim of this study is to investigate and compare mathematics teachers’ knowledge of pedagogical content, student comprehension, teaching strategies, subcomponents of pedagogical content knowledge, student errors in the relationship between perimeter and areas in rectangles, squares and parallelograms. To achieve this end, a study was carried out with 10 middle school mathematics teachers and 10 prospective mathematics teachers. In this study, the qualitative research approach of case study was used. To collect the data, an interview form consisting of four questions showing student errors related to perimeter and area was prepared. The participants were asked to comment on the questions given in the form, and their answers were recorded. Later, they were asked to write down their answers to these questions. According to the outcomes, there is a lack of student knowledge and teaching strategies, which are the subcomponents of the pedagogical content knowledge of prospective mathematics teachers. Additionally, in-service teachers' knowledge on the topics under investigation is not on a desired level. To prevent difficulties, when the concepts of "perimeter" and "area" are taught, instead of giving formulae initially, concrete materials or real-life examples about these concepts should be provided.

Keywords: pedagogical content knowledge, perimeter, area, mathematics teachers, prospective mathematics teachers

Introduction

The topics of perimeter and area are the basic subjects for the competence of elementary school teachers (Reinke, 1997). However, these are confusing topics as both involve measurement, their formulae are taught almost simultaneously to students, and memorizing these formulae may be confusing (Van de Walle, Karp & Bay Williams, 2014). This is confusing for both students and teachers. There are some studies showing that teachers often confuse the concepts of perimeter and area because they assume a constant relationship between area and perimeter (Baturo & Nason, 1996; Fuller, 1997; Heaton, 1992). According to some studies, prospective teachers have also been found to be incompetent in comprehending concepts, their knowledge is based on rules and formulae, they have some difficulties in explaining what these formulae are for, and they focus on using formulae rather than activities designed to reinforce concepts (Baturo & Nason, 1996; Berenson et al., 1997; Menon, 1998; Reinke, 1997). According to Zacharos (2006), using formulae first while measuring area leads to misconceptions about area measurement and makes it difficult to interpret the physical meaning of the numerical representation of area. According to Baturo and Nason (1996), many prospective teachers are devoid of concrete measurement experiences such as covering the surface area with measurement units, and they think of area as
“multiplication of width by length”. This situation shows that the difficulties that prospective teachers face may stem from their learning experiences at school (Baturo & Nason, 1996).

Difficulties regarding area and perimeter are usually related to conservation of area and perimeter and using inappropriate units while calculating them (not using square units while reporting area measurements) (Baturo & Nason, 1996; Ma, 1999; Murphy, 2012; Yeo, 2008). It is also generally thought that two rectangles with the same area must have the same perimeter (Van de Walle et al., 2014). However, this is not always valid. Similarly, two rectangles with the same perimeter measurements cannot be expected to have the same area, and this situation is not also limited to rectangles (Van de Walle et al., 2014).

In fact, confusion of the topics of area and perimeter, how teachers teach these topics and how they react to these errors are the basis of problems experienced by students related to area and perimeter. This is because teachers who have not exactly understood mathematical concepts cannot be expected to explain these concepts. What is more, teachers who have a good command of the subject matter but cannot present the topic in a way that students can understand also experience similar problems (Yeo, 2008). The main reason for this is that teachers use different types of knowledge for teaching mathematics (Rowland, Turner, Thwaites & Huckstep, 2009; Shulman, 1987). They must not only be equipped with a good command of the subject matter but also have the knowledge of how to present it to enhance student comprehension in the most effective way and detect students’ learning difficulties and mistakes (Gökkurt, 2014). These pieces of knowledge were first termed by Shulman (1986a) as “pedagogical content knowledge”. The concept pedagogical content knowledge is used to express teachers’ interpretations and transformations of subject-matter knowledge to support student learning. It especially involves understanding students’ learning difficulties and prejudices (Van Driel, Verloop & De Vos, 1998), and it represents certain strategies and approaches of the teacher while conveying mathematical knowledge to students (Van de Walle, 2014).

Pedagogical content knowledge may be used effectively and flexibly during the interaction process between students and teachers. Teachers’ actions while dealing with subjects largely depend on their pedagogical content knowledge (Van Driel et al., 1998). Shulman (1986b) specified teachers’ knowledge as subject matter knowledge, pedagogical content knowledge and curriculum knowledge. Shulman (1987) also pointed out that pedagogical content knowledge has two components. These are the knowledge of “representations,” which involves instructional strategies that are used by teachers to make subject matters understandable to students, and the knowledge of students’ learning difficulties, which is related to their misconceptions about subjects (Hume, 2011). Instructional strategies are the way subject matters are taught (Van Driel, Jong & Verloop, 2002), as well as representations in the forms of pictures, analogies and explanations to make a subject matter more
comprehensible for students (Shulman, 1987). Knowledge of student
comprehension, on the other hand, comprises the knowledge related to student
misconceptions, naive ideas obtained through interpretation of prior learning
experiences and preconceived ideas about the subject matter (Shulman, 1987).
Looking at difficulties related to area and perimeter, it may be seen that
these problems are not only experienced by students at schools (Livy, Muir &
Maher, 2012). In this case, it is inevitable that students often have
misconceptions about area and perimeter. To cope with this situation, therefore,
teachers should not only try to teach students in a way that avoids
misunderstandings, but they should also have approaches to deal with the
misconceptions that arise (Chick & Baker, 2005). On the other hand,
prospective teachers with a limited understanding of area may fail to help
children develop this notion because a prospective teacher’s understanding of
the nature of area is seen as a key concept in their style of teaching (Murphy,
2012). This situation also applies to in-service teachers. In order to solve the
difficulties experienced in the concepts of area and perimeter, it is important to
raise prospective and in-service teachers’ awareness on pedagogical content
knowledge related to these topics. For this reason, this study aimed to
investigate student errors in terms of the relationship between area and
perimeter in rectangles, squares and parallelograms regarding mathematics
teachers’ and prospective mathematic teachers’ in accordance with their
knowledge of students’ comprehension and teaching strategies. This study will
also reveal how the teaching experiences and mathematical backgrounds of
teachers and prospective teachers affect the knowledge of students’
comprehension and teaching strategies.

Methodology

Research Model

One of the approaches of qualitative research, case study, was employed in
this study. Since the objective was to examine the participants’ voice
recordings on their explanations and answers to questions in a detailed way, a
case study approach was used. Case studies serve to discover a phenomenon
about which little is known or to examine it thoroughly. This approach owes its
power to the researcher’s ability to investigate the case in depth and in a
detailed manner (Arthur, Waring, Coe & Hedges, 2017). In this study, the aim
is to investigate mathematics teachers’ and prospective mathematics teachers’
pedagogical content knowledge, their knowledge of student comprehension and
that of teaching strategies, subcomponents of pedagogical content knowledge,
in relation to student errors in the relationship between perimeter and area in
rectangles, squares and parallelograms in a detailed way.
Participants

This study was carried out with 10 middle school mathematics teachers (years of service between 5 and 7 years) and 10 prospective mathematics teachers who were senior students. The reason why prospective teachers were chosen from among senior students was that they were knowledgeable and competent enough in the subject matter. The in-service and prospective teachers who participated in the study were chosen on a voluntary basis. While choosing the participants, convenience sampling, which is a purposive sampling method, was utilised. The names of the teachers and prospective teachers that took part in the study are kept confidential. While the in-service teachers are coded as T1, T2, T3, etc., the prospective teachers are coded as P1, P2, P3, etc.

Data Collection

As a means of data collection, an interview form involving four questions about student errors in perimeter and area was prepared. The interview schedule was in such a way that it could be possible to determine the participants’ student knowledge and instructional strategies. If the interview questions are examined, it may be seen that the first question was prepared by inspiration from Ma’s (1999) study, while the second and third ones were inspired by Tan Şişman and Aksu’s study (2009). The fourth question was prepared by inspiration from Murphy’s (2014) question, which had been adapted from the question in Tiern et al.’s study (1990). The first question was prepared to identify student knowledge and instructional strategies about changes in the perimeter and area of a rectangle. The second question aimed to test the notion of the variability of perimeter, while the third one sought to learn about student knowledge and teaching strategies related to area conservation. The last question was prepared to identify student knowledge and instructional strategies on variability of the area and perimeter of parallelograms and rectangles. The reasons why the rectangle, the square and the parallelogram were chosen for this study were that these are the topics students are taught on the middle school level, they are interconnected, and there are very few studies in the literature dealing with the square, the rectangle and the parallelogram at the same time. The participants were given an interview form in which there were student errors. The participants’ views were recorded using a voice recorder, and they were asked to write down their answers to the questions. Similar to the case in the study by Gökkurt et al. (2015), by looking into whether the participants were able to detect student errors or not, the researchers tried to determine their “knowledge of student comprehension”, and by taking their suggestions on how to correct student errors into consideration, they tried to determine the participants’ “knowledge of instructional strategies.”
Data Analysis

Descriptive analysis techniques were employed to analyse data. In descriptive analysis, data are summarised and interpreted according to previously set themes (Yıldırım & Şimşek, 2013). For descriptive analysis, the framework prepared by Gökkurt et al. (2013) was used after making some alterations as a result of the inconsistency of the present data with the codes they used. The reason why the researchers benefited from their framework was that, while forming this framework, Gökkurt et al. (2013) formed certain draft themes and codes after collecting prospective teachers’ written answers, and they reorganised them after reading these answers repeatedly. Later, they consulted an expert on whether they were comprehensible in terms of their validity, and consequently, after certain corrections, they made them clear and understandable enough for the reader. In this study, the framework created by Gökkurt et al. (2013) was not used as the codes and themes in that study did not correspond to those in this one. Instead, in the light of the data obtained, new codes and themes were formed. An expert was consulted to check the validity of these codes and themes. When the data obtained from the participants were coded, these data were re-coded at different times by different researchers. These codes and categories are given in Table 1. The reliability percentage of the data coded according to the codes and categories in Table 1 was found to be 87%. For the uncommon codes, the researchers came together and negotiated. To ensure the validity of the study, the procedures in the study were described in a detailed way, and another researcher who is an expert in pedagogical content knowledge was consulted in the processes of preparation of the data collection tools and data analysis. Moreover, the researchers presented how they reached the results in a clear, understandable and consistent way. In the section related to the codes obtained, direct quotes taken from the participants’ answers are given. To ensure reliability, on the other hand, the researchers made sure that the results obtained were consistent with the data, and they explained the processes of data collection, forming categories and codes and analyses of these in a detailed way.
Table 1. Codes and categories

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Participants</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a partially correct solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Finding the error partially correctly</td>
<td>Finding the error partially correctly and no solution recommendations</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error partially correctly and recommending an incorrect solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finding the error partially correctly and recommending a partially correct solution</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Results

In this section, the in-service and prospective mathematics teachers’ ability to detect student errors, which is their “student knowledge”, and their suggestions, methods, techniques and strategies to correct these errors, which are called “instructional strategies,” are investigated by looking into the explanations of the participants and their written solutions. The data obtained are presented with direct quotes, and they are also given in the tables featuring categories, codes and frequencies. Table 2 shows the code, category and frequency information of the prospective teachers’ answers to the first question.
When the prospective teachers’ answers to the first question, asked in relation to the student error “If the perimeter of the rectangle increases, its area also increases” were investigated, the majority of the teachers agreed with the students. When the suggestions given by the prospective teachers who found the error correctly were examined, it was seen that only two of them offered a correct recommendation.

The following direct quote taken from the interview conducted with P10, who spotted the student error accurately and offered an accurate recommendation to correct the error, may be given as an example to the answers given by the prospective teachers.

“The student’s idea that the perimeter of the rectangle increases when its area increases is definitely wrong. The student must have thought that the lengths of the sides also increased. We can show that this is not always the case. Let us consider a rectangle with a width of 4 cm and a length of 6 cm and compare it to a rectangle with a length of 13 cm and a width of 1 cm. The perimeter of the first rectangle is 20 cm, and the perimeter of the second one is 28, which means that the perimeter has increased. When we look at the areas of these two rectangles, we see that while the area of the first one is 24 cm², the area of the second one is 13 cm².”

The ideas of P3, who could not detect the error or offer a correct solution to the students, were as follows:

“I think the way the student thought is right. Let’s think of a rectangle with a width of 2 cm and a length of 4 cm, for example. If the sides of this rectangle increase by 2 cm, the width becomes 4 cm and the length becomes 6 cm. The perimeter of the first rectangle

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**Table 2. Prospective teachers’ answers to the first question**

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Prospective Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>P7</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>P5, P10</td>
<td>2</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td>P1, P2, P6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>P3, P4, P8, P9</td>
<td>4</td>
</tr>
</tbody>
</table>
was 12 cm but now it is 20 cm, which means that it has increased.
While the area of the first one is 8 cm², that of the second one is 24 cm². So, as the area increased, the perimeter also increased.”

Table 3 shows the code, category and frequency information of the teachers’ answers to the first question.

Table 3. Teachers’ answers to the first question

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>T3, T7, T8, T9, T10</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>T1, T5</td>
<td>2</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td>T4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>T2, T6</td>
<td>2</td>
</tr>
</tbody>
</table>

As seen in Table 3, seven of the mathematics teachers found the error correctly while answering the same question. In Table 3, one may see that two of the teachers recommended a correct solution to correct the student’s error.

To illustrate the answers given by the teachers, the answer given by T1, who found the student error correctly and recommended an accurate solution to correct it, may be quoted as follows:

“The student thought wrong. We can’t generalize this situation. Let me show that it can be wrong with an example:

Figure 1. Answer of T1

The opinions of T6, who could not find the error correctly or come up with
a recommendation for the students, were as follows:

“I also think the student thought rightly, because the area increases when the perimeter increases. Take a rectangle with a width of 4 cm and a length of 7 cm, for example. Suppose that we have another rectangle with a width of 4 cm and a length of 10 cm. While the perimeter of the first rectangle is 22 cm, and its area is 28 cm², the perimeter of the second one is 28 cm, and its area is 40 cm².”

Table 4 shows the code, category and frequency information of the prospective teachers’ answers to the second question.

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Prospective Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>P2, P3, P5, P8, P10</td>
<td>5</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td>P4, P6, P9</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>P1, P7</td>
<td>2</td>
</tr>
</tbody>
</table>

The prospective teachers had difficulty in understanding the third question and spent a lot of time in forming the parallelogram. Some of them even failed to form one.

The response given by P6, who was one of the prospective teachers who identified the student error correctly and offered an adequate solution, was as follows:

Figure 2. Answer of P6
He thought that when the shapes changed, the sides would decrease, and thus, the perimeter would also decrease, or maybe, he was confused by $\sqrt{5}$ while making the calculations with square root expressions. The first shape is a square and its perimeter is $8a$. The perimeter of the second shape is, on the other hand, $4a + 2a\sqrt{5}$. Let’s have a look at the range of $\sqrt{5}$. It is closer to 2, and this makes it more than the perimeter of the square. Here, the child knows that this length is the hypotenuse length, and this is the longest edge.

The response of P1, who identified the student error incorrectly and could not offer a correct recommendation, was as follows:

“The student’s answer is incorrect. We can’t make a difference in the length of the shape by adding the cut-up part to other parts of the shape. We can explain it to the student by drawing it.”

P4, who had no ideas as to why the student gave a wrong answer, provided the following response:

“...but how will it be possible? We can’t place the triangles. The triangles have a right angle. How can I make a parallelogram with them?”

Table 5 shows the code, category and frequency information of the teachers’ answers to the second question.

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>T1, T8</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>T2, T3, T5, T9, T10</td>
<td>5</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td>T7</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>T4, T6</td>
<td>2</td>
</tr>
</tbody>
</table>

When the teachers’ answers to the third question were examined, it was clear that seven of the teachers identified the student error correctly, two of them failed to do so, and one of them did not make any comments regarding it.

The response of T5, who identified the student error correctly and gave an
adequate recommendation, was as follows:

“Let me draw a square whose length is $2a$ cm. Its perimeter is $8a$ cm. Then let’s form the expected shape. It’s $8+2\sqrt{5}$. That is to say, the perimeter has increased.

Figure 3. Answer of T5

[Diagram of a square and a modified shape]

Here, the student gave a wrong answer. He thought that the shape is narrow. We can show him how it is with the help of the Pythagorean theorem. ...because if the sides are considered as the hypotenuse of the triangle, they have increased, and they became $a\sqrt{5}$.”

T4, who identified the student error incorrectly and could not offer an accurate recommendation, gave the following answer:

“Let me draw it and see.

Figure 4. Answer of T4

[Diagram of a parallelogram and a square]

The perimeter of the parallelogram is less than that of the square.”

Table 6 shows the code, category and frequency information of the prospective teachers’ answers to the third question.
Table 6. Prospective teachers' answers to the third question

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Prospective Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>P3, P5, P8</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>P1, P2, P4, P9, P10</td>
<td>5</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>P6, P7</td>
<td>2</td>
</tr>
</tbody>
</table>

In the third question, eight of the prospective teachers correctly identified the error. The opinions of P10, who spotted the error correctly and offered an accurate recommendation, were as follows:

Figure 5. Answer of P10

“He made a mistake. He may have thought that there would be more sides. Confusing the area with the perimeter, he thought as if the number of the sides increased, and so did the perimeters. The area doesn’t change if the shapes are shifted or placed somewhere else. ...because the area is the surface covered by the shape. As the total area doesn’t change, the areas are equal to each other. For instance, when you cover the floor of a rectangular room completely with a carpet, no matter how much you cut it into smaller pieces and place them side by side, the total [area] will definitely not change.”

The ideas of P7, who thought the same way as the student and thus identified the student error incorrectly, were as follows:
Figure 6. Answer of P7

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This part has been cut.

Let’s take it here.
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“I think his answer is right. ...because the second shape covers a larger area. There are zigzags in the second one, so the area of the first shape is smaller.”

Based on these explanations, it may be stated that P7 did not know that the area would not change when the shape is shifted or placed somewhere else.

Table 7 shows the code, category and frequency information of the teachers’ answers to the third question.

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Finding the error correctly but no solution recommendations</td>
<td>T4, T8</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>T2, T3, T5, T6, T7, T9, T10</td>
<td>7</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td>T1</td>
<td>1</td>
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</table>

It was observed that the majority of the teachers identified the error correctly, and only one of them did not even do any reasoning related to it.

The following were the opinions of T10, who spotted the error correctly and recommended an appropriate solution:

“The student thought wrong. The student might have confused the area with the perimeter here. It says zigzag here. Because it is indicated this way, most probably he thought that when the perimeter increases, the area will also increase. Additionally, he thought that the area was larger as there were two more parts. Based on the definition of area, I try to eliminate the misconception between area and perimeter with concrete examples. For instance, I show the area covered by sugar cubes as a whole, and then, I leave the sugar cubes in a different place. I have the area of each sugar
cube calculated. Therefore, the students will notice that the area has not changed.”

Lastly, the fourth question will be examined at two stages. First of all, the findings related to how the teachers commented on the change in the perimeters of the two shapes with equal areas (Shapes 1 and 2), and secondly, how they viewed the change in the areas of the two shapes with equal perimeters (Shapes 1 and 3) will be given.

Table 8 shows the code, category and frequency information of the prospective teachers’ answers to the fourth question with respect to how the perimeters of two shapes with equal areas change.

**Table 8. Prospective teachers’ answers about how the perimeters of shapes change**

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Prospective Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
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<td>P1, P3, P9, P10</td>
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<tr>
<td>Not finding the error correctly</td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td>P8</td>
<td>1</td>
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<tr>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>P2, P6, P7</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Finding the error partially correctly</td>
<td>Finding the error partially correctly and recommending an incorrect solution</td>
<td>P4</td>
<td>1</td>
</tr>
</tbody>
</table>

When the responses given by the prospective teachers were examined, it was seen that four of the prospective teachers could not identify the student error correctly.

The ideas of P16, who identified the student error correctly and offered a correct suggestion, were as below:

“The areas of the first and second shapes are equal. The student couldn’t figure out how the sides of the parallelogram would change while comparing their perimeters. Probably, he was inclined to find a relevant formula as the length of the vertical edge leg was given as 4 cm, but the lengths of the side legs were not given. The length of the base side was given as 9 cm, and because the lengths of the height and the side legs were not given, the student automatically
thought there was missing information. To correct the error, I would emphasize that knowing the side lengths is not essential. I would remind the student that the side lengths of the second shape would be longer than 4 cm because of the Pythagorean theorem. That is, the perimeter would measure longer.”

The following were the views of P6, who failed to identify the student error as she shared the same opinion as the student and thus offered an incorrect suggestion:

“In the second shape, the side lengths are not given. Oh no, I can’t find the perimeter. I think there is something missing. There is missing information in the base length as well... I can’t say anything. In fact, I guess the student thought right.”

The ideas of P4, who identified the error partially correctly but could not recommend a correct solution, were as follows:

“I think the student thought wrong. A triangle is formed in the second shape, and the student could not find a connection between the side of the triangle and the parallelogram. In fact, no other information is needed.”

Table 9 shows the code, category and frequency information of the teachers’ answers to the fourth question with respect to how the perimeters of two shapes with equal areas change.

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Finding the error correctly but no solution recommendations</td>
<td>T3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>T1, T5, T7, T10</td>
<td>4</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td>T4, T6, T8</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td>T2</td>
<td>1</td>
</tr>
<tr>
<td>Finding the error partially correctly</td>
<td>Finding the error partially correctly and no solution recommendations</td>
<td>T9</td>
<td>1</td>
</tr>
</tbody>
</table>
When the explanations of the teachers on the student’s answer with respect to how the perimeters of two shapes with equal areas (1 and 2) change were considered, three of the teachers could not make any comments regarding this question. Only one of the teachers had the same opinion as the student.

T5, who spotted the student error accurately and offered a correct recommendation, expressed the following opinions:

“He looked at the first and the second shapes. There is no missing information; the student is wrong. I would tell him that, when we compare the first and second shapes, the length of the side edge of the second shape is longer. While doing it, I would also remind him of the hypotenuse. In the second shape, if the vertical edge is 4 cm, because of the property of the hypotenuse, the side edges must be longer than 4.”

T9, who partially identified the error but could not recommend a correct solution, expressed the following ideas:

“The areas of the first and the second shapes are equal. When we compare their perimeters, we see that the student thought wrong. ...because the lengths of the short sides of the parallelogram are different than those of a rectangle, their perimeters are also not equal to each other.”

As it may be seen, even though T9 mentioned the existence of student error, she did not point out what this error stemmed from.

The answer given by T2, who identified the error in an incorrect way, was as follows:

“The student gave a wrong answer. ...because the shapes with the same areas are 1 and 2. When their perimeters are taken into account, while the perimeter of the rectangle is 26, that of the parallelogram is smaller than 26.”

In the fourth question, the teachers were also asked to comment on how the areas of two shapes with equal perimeters (Shapes 1 and 3) changed. The findings related to this were as follows.

Table 10 shows the code, category and frequency information of the prospective teachers’ answers to the fourth question with respect to how the areas of two shapes with equal perimeters change.
Table 10. Prospective teachers’ answers about how the areas of shapes change

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Prospective Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>P3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>P1, P5, P6, P8, P10</td>
<td>5</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>P2, P7, P9</td>
<td>3</td>
</tr>
<tr>
<td>Finding the error partially correctly</td>
<td>Finding the error partially correctly and recommending an incorrect solution</td>
<td>P4</td>
<td>1</td>
</tr>
</tbody>
</table>

When the prospective teachers’ answers to the question related to the student error in finding the areas of shapes with the same perimeters were analysed, it may be noted that most of the prospective teachers correctly identified the student error.

The following was the answer given by P6, who spotted the error correctly and provided a correct recommendation:

“Now, we see that the perimeters of the first and the third shapes are equal. Here, the student must have thought that he could not calculate the area as the height was not given. …but if he had drawn a vertical line starting from the corner, he could have found it, indeed. When we form a triangle by taking the vertical edge as the height and the short edge as the hypothenuse, the height is supposed to be smaller than the hypothenuse that is 4. …as the area of the third shape will be less than 36, that is, smaller than the area of the first shape.”

The statements of P9, who could not identify the error as he thought the same way as the student, may be quoted as follows:

“The student will compare one to three. Now, the third shape is a parallelogram, and its base length is 9 cm, but its short edge is 4 cm, and these edges do not intersect vertically. I think the student said it right. …because the [height of the] first shape and the height of this shape will turn out to be different, and their areas will also be different. …but as we don’t have the necessary information, we can’t
say anything about it now. Some more information should have been
given.”

The response given by P4, who disagreed with the student but still partially
identified the error and provided incorrect recommendations, is quoted below:

“The first and the third shapes must be dealt with. I think what he
said is wrong. First of all, when we look at them, the perimeters [of
the first and third shapes] may be equal, but the area will change in
parallel with the shape. The reason for this is that the area
calculations for shapes such as the square, the rectangle and the
triangle are different.”

Table 11 shows the code, category and frequency information of the
teachers’ answers to the fourth question with respect to how the areas of two
shapes with equal perimeters change.

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Teachers</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finding the error correctly</td>
<td>Finding the error correctly but no solution recommendations</td>
<td>T3, T4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Finding the error correctly and recommending a correct solution</td>
<td>T1, T5, T7</td>
<td>3</td>
</tr>
<tr>
<td>Not finding the error correctly</td>
<td>No answer</td>
<td>T8</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and no solution recommendations</td>
<td>T2, T10</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Finding the error incorrectly and recommending an incorrect solution</td>
<td>T6</td>
<td>1</td>
</tr>
<tr>
<td>Finding the error partially correctly</td>
<td>Finding the error partially correctly and recommending a partially correct solution</td>
<td>T9</td>
<td>1</td>
</tr>
</tbody>
</table>

While half of the teachers identified the error correctly, the rest of them
could not do it. When the findings were analysed, it was found that the teachers
had the most difficulty with the fourth question as not all the side lengths of the
parallelogram were given, and neither were the angles.

T1, who identified the student error correctly and offered correct
recommendations, stated the following ideas:

“The student wants to be given everything. When he compares the first
and the third shapes, he thinks that since the height is not given in the
third shape, he cannot calculate the area either. The student was not
asked to calculate the area anyway. He was asked to make a comparison.
To show it to the student, I would show him that the area of the rectangle
is larger. If the short side of the parallelogram is 4 cm, the height of the
long side will be less than 4 cm. …because there is the hypotenuse.”

T6, who wrongly identified the error of the student and offered incorrect
recommendations, stated the following ideas:

“The perimeters of one and three are equal. When we compare their
areas, we see that the student gave a correct answer as the height is
unknown in the third shape. Concerning this, even if the height is not
given, if the angle had been given, we could have done something.”

The ideas of T9, who partially accepted the wrong answer as correct and
partially gave a correct recommendation, were as below:

“In fact, there is no need for additional information. As the sides of
the first and third shapes are equal to each other, their perimeters
are equal, as well. …but as the width of the rectangle and the height
which is used in the area calculations of the parallelogram in the
third shape are different, their areas will also be different.”

Discussion

This study aimed to investigate the pedagogical knowledge of student
errors in terms of the relationship between area and perimeter in rectangles,
squares and parallelograms by mathematics teachers and prospective
mathematics teachers in accordance with the knowledge of students’
understanding and teaching strategies.

When the responses of the participants on the student misconception “As
the perimeter of the rectangle increases, its area also increases” were analysed,
it was observed that most prospective teachers thought the same way as the
student. Although the teachers generally detected the error correctly, some of
them were also found to detect the error incorrectly. In their study looking into
the misconception that as the perimeter of the rectangle increases, its area also
increases, Ma (1999) established that teachers had the same misconception as
students. Livy et al. (2012) and Wanner (2019) also found that prospective
teachers had the same mistakes as students. When the recommendations of the
participants regarding the student error were examined, it was seen that the in-
service and prospective teachers made recommendations by giving the
rectangular examples where the areas decreased, despite the increase in the
perimeters. Consequently, according to the answers given in response to the
first question, the mathematics teachers seemed to have more student
knowledge and instructional strategies than the prospective teachers. As
regards to this, Menon (1998) also pointed out that the pedagogical content
knowledge of teachers developed in time. As Menon put it, teachers can teach
certain concepts better, highlight the connections between subjects better and
conceive better examples in time. When a parallelogram was created from a
square, it was seen that the teachers perceived it more correctly compared to the
pre-service teachers in terms of the student error regarding how the perimeter
lengths changed. This finding was in parallel with that in Tan Şişman and
Aksu’s study (2009), although their subjects were seventh graders. Their study
revealed that seventh graders did not believe that the perimeter of the shape
would change when a new shape was formed after it was cut into small pieces
and reassembled using exactly the same pieces.

In the third question, the student was asked to form a new shape out of a
rectangular sheet after it was cut downwards starting from its long side with
zigzags by placing the cut-out part below the rectangle and then to compare the
areas of the first and the second shapes. In this case, some prospective teachers
stated that they increased the area because the second form took up more space.
Baturo and Nason (1996) also found that prospective teachers had limited
subject matter knowledge, and they were unaware of the fact that even if cut
into separate parts, a two-dimensional area of a shape would stay the same
when the same parts are reassembled forming a different shape. Here, in fact,
the learners confuse the concepts of perimeter and area and the fact that they
have misperceptions related to area conservation. Although conducted with
seventh graders, regarding this, according to Marshall (1997), 7th grade
students had a strong understanding of the concept of perimeter, but their
understanding of the concept of area was not well-developed, and in his study,
he found that the relationship between area and perimeter could not be
understood. Tan Şişman and Aksu’s study (2009) also supported these findings
in that seventh graders did not have a conception of area conservation.

In the fourth one, the first subject in question was the misconception that
the perimeters of the parallelogram and the rectangle cannot be compared when
all sides are not known. Some of the prospective teachers thought that, as there
were missing sides, nothing could be said about the length of one side of the
parallelogram. Some of the teachers, on the other hand, made a mistake in
comparing the perimeters as one of the sides of the parallelogram was not
given.

The second subject in question four was the misconception that the areas
of parallelograms and rectangles whose perimeters are equal cannot be
compared due to the fact that all the side lengths are not known. Some teachers
and prospective teachers were revealed to agree with the student that, since the
height was not given, a comparison could not be made, as there was missing
information. When the data obtained were examined, the participants were
found to have the most difficulty answering the fourth question as the angles
and all the side lengths were not given. In her study, in order to expose
prospective teachers’ subject matter knowledge, Murphy (2012) asked them to
compare the perimeters of shapes with equal areas and the areas of shapes with
equal perimeters. Although the two studies were similar in that, in both studies, prospective teachers were found to have difficulty finding the area of the parallelogram, this study was different from the other one as it was also found in this study that the prospective teachers compared the areas and perimeters by placing two shapes on each other. In her study, Herendiné-Kónya (2015) showed students at the ages of 7-11 two parallelograms with the same lengths but different areas, and wanted the students to compare the areas of these parallelograms to the same edges, and it was found that, although it was easy for them to see that the areas were different, the students claimed that the area of these two parallelograms was the same.

Conclusions

Overall, when the answers given in response to the interview questions are taken into account, it may be stated that, even though they are on different levels of learning, neither seventh graders nor teachers or prospective teachers have proper conceptual learning. The participants were observed to have the most difficulties in questions related to parallelograms. Marchis (2012) also established in her study that prospective teachers were less successful in tasks related to parallelograms. In this study, the prospective and in-service teachers had a tendency to make a calculation on the area and the perimeter, but they had hardships explaining the concepts. This situation may have stemmed from the teachers and prospective teachers focusing on computational knowledge rather than conceptual knowledge. Similar to these findings, Menon’s study (1998), which was conducted to investigate middle school teachers’ understanding of perimeter and area in triangles and rectangles, also maintained that they had a computational understanding of them rather than a conceptual one. In their studies on the subject matter knowledge of prospective teachers, Berenson et al. (1997), on the other hand, established that many of them had computational knowledge. According to Livy et al. (2012), this computational knowledge limits students’ development of conceptual understanding and the potential of instructional strategies.

The results of this study showed that, while the mathematics teachers need considerable improvement in student knowledge and the knowledge of teaching strategies, which are subcomponents of pedagogical content knowledge, the prospective teachers have not reached the required level of student knowledge and that of teaching strategies. Moreover, in comparison to the mathematics teachers, the prospective teachers were found to be incompetent with respect to the required mathematical subject matter knowledge. Livy et al. (2012) also noticed that prospective teachers had similar strengths and weaknesses as regards to their subject matter knowledge and pedagogical content knowledge related to area and perimeter, and they stated that prospective teachers had deficiencies in subject matter knowledge, which is necessary for them to understand area and perimeter and make a connection between them.
Although prospective teachers take teaching classes during their undergraduate education, the instructional explanations they made in the study were not sufficient. In this respect, teaching classes may be revised both theoretically and practically to prepare prospective teachers as aimed by the mathematics curriculum. Besides, in future studies, not only their pedagogical content knowledge but their subject matter knowledge could also be investigated. When “perimeter” and “area” are first taught, teachers should teach students what these two concepts are through concrete materials or real-life examples instead of giving them formulae. It is possible to comprehend the perimeter-area relationship by presenting different geometric shapes consisting of the combination of the same number of unit cubes. Moreover, as pointed out by Tan Şişman and Aksu (2009), teachers could help reinforce the understanding of the changes in perimeter and area conservation through activities at school involving cutting, folding and reassembling.

References

Prague: Charles University in Prague, Faculty of Education and ERME. 
https://hal.archives-ouvertes.fr/hal-01287005
Appendix

Dear participants;
Please answer the following questions sincerely so that we can figure out your opinions of them. The length of our interview will be approximately 40 minutes. In this study, your identity will be kept confidential. Thank you for your participation.

Interview Questions

1) One of the students said, “If the circumference of the rectangle increases, its area also increases.” Do you think this statement is correct? What would your answer be? Explain.

2) Form two rectangles by cutting a square sheet of paper into two equal parts. Cut one of these rectangles diagonally into two identical parts. Using all of the shapes that you have obtained (2 triangles and 1 rectangle), form a parallelogram. How did the area and the perimeter measurements of the first shape and the newly formed shape change? Explain.”

   In response to this question, one student wrote “The perimeter of the parallelogram is less than that of the square.”
   • Do you think what the student says is right?
   • Why might this student have thought in this way?
   • Were you to encounter such a situation, how would you react in response to such an explanation?

3) After cutting a rectangular sheet of paper downwards starting from its long side with zig-zags, form a new shape by placing the cut-out part below the rectangle. How different are the areas of the first shape and the newly formed shape? Explain.” In response to this question, one of the students said, “The area of the second shape is bigger than that of the first shape.”
   • Do you think what the student says is right?
   • Why might this student have thought in this way?
4) The students were asked to explain i) how the perimeters of the shapes with identical areas have changed and ii) how the areas of shapes with the same perimeters have changed by looking at these shapes. One of the students said that these shapes could not be compared, for not all of the side measurements are known.

1. 2. 3.

9 cm 4 cm 9 cm 4 cm 9 cm 4 cm

4) Were you to encounter such a situation, how would you react in response to such an explanation?

7 8

Do you think the way this student thought is correct?

9 Why might this student have thought in this way?

10 Were you to encounter such a situation, how would you react in response to such an explanation?