Infinity: A Number or not a Number? Definitions and Images for the "Infinity" Concept

This study is a part of longer study aimed to examine whether pre-service teachers (hence for: PST) understand if the infinity concept is a number or not a number, during examining the concept definitions and the concept images, as well as the relation between them.59 PST were involved in this study. A questionnaire was designed to explore the cognitive schemes of the infinity concept that are evoked by the PST during numerical tasks. One question aimed to check whether the PST knew to define the infinity concept. Five others were designed to categorize how PST works with the infinity concept and how this related to the definition. The results show that only 10% of our sample knew the formal definition. Only between 27% - 50% of our sample knows that "infinity is not a number". The results show also that 71% and 7% of our sample failed with the misconceptions $0 \cdot \infty = 0$ *and* $\infty - \infty = 0$ *(respectively). The reasoning and argumentations that the PST gave to decide whether an expression is a number or not a number were intuitive and not related to the formal definition of the concept.*

Keywords: *Infinity in Mathematics. Concept Definitions And Concept Images.*

Introduction

The Infinity concept and it is use occupies a central place in mathematics curriculum in schools, colleges and universities. The infinity concept used in different domains in mathematics: pre-calculus, calculus, set theory, algebra and geometry. In many countries, these domains are taught in middle school (functions, algebra and geometry) and mentioned intensively again and again in high school (calculus, algebra and geometry), academic colleges and universities (calculus, algebra, and set theory). In most set theory text books, one can find the following (Stewart and Tall, 1977) Proposition (Cantor):
"If a set B is *infinite*, then there exists a proper subset $A \underset{\neq}{\subset} B$ and a bijection $f : B \to A$."

In most calculus or advanced calculus mathematical text books one can find definitions for the infinity concept ($+\infty$), such as the following (Ayres and Mendelson, 1992):

"We say that a sequence $\{S_n\}$ *approaches to* $+\infty$*, and we write* $S_n \to +\infty$ *or* $\lim_{n\to\infty} S_n = +\infty$ *if the values* S_n *eventually become and thereafter remain greater than any pre assigned positive number".*

The correspondent notion for functions is the (Ayres and Mendelson, 1992) following:

"We say that $f(x)$ *approaches* $+\infty$ *as x approaches to a, and we write* $\lim_{x\to a} f(x) = +\infty$*, if x approaches to its limit a (without assuming the value a),*

$f(x)$ eventually becomes and thereafter remains greater than any pre-assigned positive number, however large".

According to the correct theory, and during the chapter of fundamentals of elementary calculus, the theorem on limits of sums, products and quotients written and explained in detail including many examples and exercises.

Later, and according to ***L'HOSPITAL'S rule***, expressions like $(1-\cos x)\cdot\cot x$ or $x\cdot\ln x$ when $x=0$ treated in detail during examples, exercises and even in exams. These expressions are undefined when $x=0$ but there limits are well defined when $x\to 0$. (For instance: $\lim\limits_{x\to 0}(x\cdot\ln x)=\lim\limits_{x\to 0+}\frac{\ln x}{\frac{1}{x}}=\lim\limits_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim\limits_{x\to 0-}\frac{x^2}{x}=\lim\limits_{x\to 0+}x=0$). Before operating ***L'HOSPITAL'S rule*** the learner should have the knowledge that the expression $x\cdot\ln x$ is undefined when $x=0$. This piece of knowledge should be in the mind of the learner before deciding whether to operate L'HOSPITAL'S rule or not. It means that; before using L'HOSPITAL'S rule the learner should have in his or her mind the simple, but important pieces of knowledge that expressions like $\infty-a,\ a\in R$ or, $0\times\infty$ or, $\infty-\infty$ or, a^{∞} for $a\rangle 1$, are undefined expressions. In order to have these pieces of knowledge, the learner should have another pieces of knowledge; the *numerical aspects* of the ***infinity concept***. By numerical aspects we mean the definition of the concept and the numerical properties of the concept. In our case the properties: $\infty-a,\ a\in R$ or, $0\times\infty$ or, $\infty-\infty$ or, a^{∞} for $a\rangle 1$ if each of them is a numbers whether not.

Sometimes, in order to present a new concept, authors of mathematical textbooks (calculus or advanced calculus) don't mention important properties of the concept. Properties like $\infty-a,\ a\in R$ or, $0\times\infty$ or, $\infty-\infty$ or, a^{∞} for $a\rangle 1$, not mentioned or rarely mentioned in these textbooks. According to the correct theory, the definition and the use of the infinity properties mentioned in algebra and modern algebra courses. The *correct theory* of *infinity* and its *properties* use mentioned again and again in the case of equations solution or systems of linear solution.

From mathematical correct theory point of view, the above definitions and rules are formal, rigorous and general.

In the opening of the chapter about Cardinal numbers (Stewart and Tall, 1977, p.298), the authors wrote:

"*'WHAT IS INFINITY?' When some first year university students were asked to this question recently, the consensus was 'something bigger than any natural number'. In a precise sense, this is correct; one of the triumphs of set theory is that the concept of infinity can be given a clear interpretation. We find not one infinity, but many, a vast hierarchy of infinities …"*

Not once, during my experience as a mathematics teacher, I meet students in high schools or PST in academic colleges or universities, who have difficulties on the understanding of the notion of infinity, during their use of the concept in solving problems activities; especially with the concept definition and with the explanation of arguments related to the concept.

Meaningful learning in mathematics, so we believe, is to use the ***correct theory***, i.e. it is to internalize the mathematical concepts definitions and the properties related to these concepts definitions. Otherwise, it becomes ***meaningless*** mathematics.

Theoritical Background

The research includes many studies on the learners' misunderstanding of concepts related to the infinity concept. It was founded that, the definition recognition of the relevant concepts does not ensure proper and correct use of such concepts. Vinner and his colleagues claimed that a possible explanation of this phenomenon is that: during problem solving activities, the learner based their solutions on ***concept images*** and not on the ***concept definition*** (Tall & Vinner, 1981; Vinner & Hershkowitz, 1980, 1981; Vinner,1982, 1983, 1991; Vinner & Dreyfus, 1989; Rasslan & Vinner, 1997, 1998; Rasslan and Tall, 2002; Tsamir & Rasslan & Dreyfus, 2006).

According to Tall and Vinner (1981) concept image is all mental pictures, properties and processes associated with the concept in the learner mind.

In the literature, the understanding of infinity is associated with two different concepts- the concept of *potential infinity* and the concept of *actual infinity* (Dubinsky, Weller, McDonald & Brown, 2005; Fishbein, Tirosh & Hess, 1979; Fishbein, 2001, Hannula & Penkonnen, 2006; Hannula, Maijala & Soro, 2006; Monaghan, 2001; Moore, 1995; Moreno & Waldegg, 1991; Tall, 2001). In the current research we found much research that deals extensively with the differences between the two concepts and the ways they *complement each other* from mathematics, philosophical, and even from historical (from Aristotle (322-384 BC, Galilei, until Bolzano (1781-1848) points of view (Kolar and Cadez, 2012). For this reason we will introduce the two concepts and the ways they *complement each other* briefly.

Potential infinity is related to an ongoing process without an end, the end for example, counting the natural numbers *1, 2, 3, 4, …*. It is an infinite process of which neither the end nor the last term (of the sequence) can be determined. We can imagine the procedure of acquiring ever new numbers, but we cannot realize it in practice.

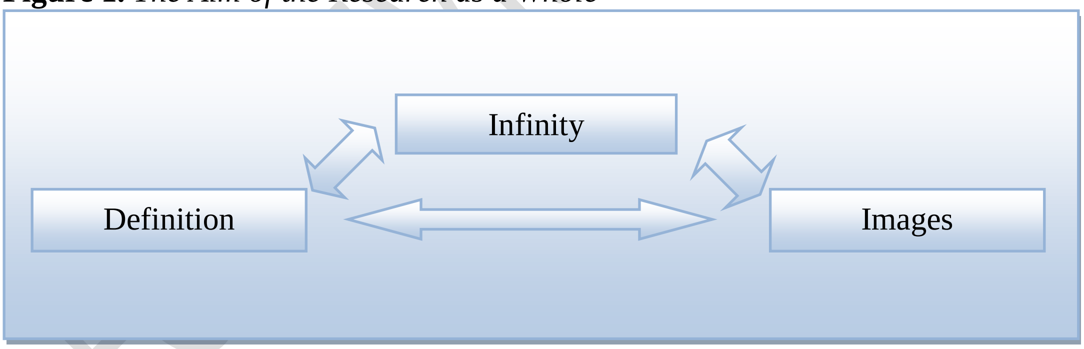
On the other hand, the concept of *actual infinity* attributes a finite entity to this infinite process. We could say that actual infinity defines the state in infinity, whereas potential infinity defines the process that creates infinite sets. According to Fischbein (2001, p. 310):

"…the potential infinity is not an existing, a given infinity. We cannot conceive the entire set of natural numbers, but we can conceive the idea that after every natural number, no matter how big, there is another natural number."

According to Kolar and Cadez (Kolar and Cadez, 2012), there are a discussion of the understanding of infinity in children, teachers and primary teacher students. It focuses on a number of difficulties that people cope with when dealing with problems related to infinity such as its abstract nature, understanding of infinity as an ongoing process which never ends, understanding of infinity as a set of an infinite number of elements and understanding of well-known paradoxes with the aim of researching their understanding of the concept of infinity. The focus was on findings out how primary teacher students who received no in-depth instruction on abstract mathematical content understand different types of infinity: *infinitely large*, *infinitely many* and *infinitely close*, what argumentation they provide for their answers to problems on infinity and what their misunderstandings about infinity are. The results show that the respondents' understanding of infinity depends on the type of the task and on the context of the task. The respondents' justifications for the solutions are based both on actual and potential infinity. When solving tasks of the type 'infinity large' and 'infinity many' they provide justifications based on actual infinity. When solving tasks of the type 'infinity close', they use arguments based on potential infinity. The conclusion of their research was that when the respondents feel unsure of themselves, they resort to their primary method of dealing with infinity, that is, to potential infinity.

Figure 1 is a schematic presentation of the overall goal of the research. In this paper, we exemplify the first stage of the study: is the 'infinity' a number or not a number? as well as the definitions and images of the infinity concept.

Figure 1. *The Aim of the Research as a Whole*



This study investigated the following:

1. What are the common definitions of the infinity concept given by PST?
2. What are the main images of the infinity concept that these PST use in identification tasks?
3. What are the main misconceptions that these PST that these students have according to infinity concept?
4. What kind of reasoning does PST give when they decide if a certain expression is a number or not number?

Method

Sample

Our sample comprised 59 PST in three classes (30, 21 and 8 PST) in three academic colleges (2 in the center and 1 in the north) in Israel. Their majority is mathematics /computer and sciences/mathematics (39 and 20 respectively). All the PST in our sample, already, finishes successfully academic courses of ***set theory***, ***linear algebra, geometry*** and ***calculus*** in their colleges.

The Questionnaire

The questionnaire (Figure 2) had two main questions: Question 1 was designed to examine the ability of PST to decide and explain if each of the five expressions: ∞, $0\times\infty$, $\infty - a$, if $a \rangle 0$, $\infty - \infty$ and a^{∞}, if $a \rangle 1$, is a number or not a number, whereas; Question 2 designed to examine their definitions to the infinity concept. The questionnaire was administered to all subjects in the sample.

Procedure

The questionnaire was administered to the PST in their classes. They were not asked to fill in their names, only their background information. It took them about 50-60 minutes at most to complete the questionnaire. All the questions in the questionnaire were analyzed in details by the authors in order to determine the answers' categories.

Figure 2. The Questionnaire

1. Which of the following expressions A - E: represent "**a number**" and which of them represents "**not a number**"? <u>Explain</u> your answer!
   A. ∞
   B. $0\times\infty$,
   C. $\infty - 2^{100}$
   D. $\infty - \infty$
   E. 2^{∞}
2. In your opinion, what is "**Infinity**"?

Results

The Definition Categories

We categorized the PST`s answers according to methods described elsewhere (Rasslan and Tall, 2002; Rasslan and Vinner, 1997). We illustrate each category with a number of sample responses.

Question 2: In your opinion, what is "Infinity"?

Table 1. *Distribution for the Definition Categories in Question 2. (N = 59)*

Category	Examples	Distribution Number (%)
I. For any number we choose, we can found a number greater than of it.	*For every great number we can found a number greater of it.*	**6 (10%)**
II. A large number.	*Very large number.*	**11 (19%)**
III. Unlimited something	*Means, not certain something.*	**7 (12%)**
IV. Represents infinity of numbers or represents infinity of integers.	*∞ Represents infinity of numbers. Represents infinity of integers.*	**13 (22%)**
V. Not a number.	*Infinity is not a number*	**4 (7%)**
VI. Nonsensical expressions	*means that there is no elements.*	**16 (27%)**
VII. No answers.		**2 (3%)**

In the above categorization (Table 1) only 10% of our PST sample (categories I) give an accepted formal definition of the infinity concept. The majority 87% of our sample (categories II, III, IV, V, VI and VII) give erroneous responses. The PST in our sample, were not directed to memorize definitions and the majority do not appear to be able (or willing) to explain the definition of the infinity concept.

The Concept Images. Questions 1 (A – E)

Various aspects of the infinity concept, as conceived by the PST, were expressed in their answers to questions 1.A to 1.E. Some of these aspects are given below:

Question 1. A. (∞)

Table 2. *Distribution for the Categories in Question 1. A. (N = 59)*

Category	*Examples*	Distribution Number (%)
I . Right answer: The largest number Of all the Real Numbers.	*Not a number. The intention of infinity is the largest number founded in R .*	**1 (2%)**
II. Right answer. The whole Numbers.	*It is not number. Whole numbers.*	**15 (25%)**
III. Right answer. Not a certain Value. Or, not a certain number.	*Not a number. It is not certain value.*	**7 (12%)**
IV. Right answer. Ritual repeating Of the Infinity concept.	*Not a number. Infinity.*	**6 (10%)**
V. Right answer without reasoning.	Not a number.	**9 (15%)**
VI. Wrong answers without Reasoning. Or, wrong answers With nonsensical reasoning.	*1. A number, but it is very large.* *2. A number which contains infinity of solutions.*	**21 (36%)**

From the above categorization (Table 2) it turns out that 49% (categories: I, II, III, and IV) of our PST sample give right answers. For other 15% of our sample (category V) we cannot claim the above, but we cannot claim the opposite. The remaining 36% of our sample (category IV) respond wrongly.

Question 1. B. ($0\times\infty$)

Table 3. *Distribution for the Categories in Question 1. B. (N = 59)*

Category	*Examples*	**Distribution Number (%)**
I . Right answer. An undefined Expression.	*1. Not number. An Undefined expression.*	**10 (17%)**
II. Right answer without reasoning.	*Not a number.*	**6 (10%)**
III. The misconception: $0\times\infty=0$.	*1. A number* $0\times\infty=0$. *The 0 is a number.*	**42 (71%)**
IV. Wrong answer without Reasoning.	*A number.*	**1 (2%)**
V. No answer.		

From the above categorization (Table 3) it turns out that only 27% (categories I, II) of the PST sample gives right answer. For other 2% (category IV) of our sample we cannot claim the above but we cannot claim the opposite. The majority of our sample 71% (category III) gives wrong answer. The huge number (71%) of PST in Category III fall with the misconception that $0\times\infty=0$, because they thought that ∞ is a number and then they operate the rule $0\times a=0$ for all $a\in R$.

Question 1. C. ($\infty-2^{100}$)

Table 4. *Distribution for the Categories in Question 1. C. (N = 59)*

Category	*Examples*	**Distribution Number (%)**
I. Right answer. Forbidden Operation. Or, $\infty-2^{100}=\infty$.	*Not number.* $\infty-2^{100}=\infty$ *And* ∞ *is not a number.*	**30 (50%)**
II. Right answer, without reasoning.	Not number.	**8 (14%)**
III. Wrong answer without reasoning or answers (wrong or right) based on nonsensical Argumentations.	*1. A number. Very large number.* *2. Not number. It is expression.* *3. A number.*	**14 (24%)**
IV. The misconception: the difference between two numbers is a number.	*A number. The subtraction of two numbers is a number.*	**4 (7%)**
V. No answer.		**3 (5%)**

From the above categorization (Table 4) it turns out that 50% (category I) of our PST sample (category I) give right answers. For other 14% (category II) we cannot claim the above but we cannot claim the opposite. The remainder 36% (categories III, IV, and V) gives wrong or nonsensical answers.

Question 1.D. ($\infty - \infty$)

Table 5. *Distribution for the Categories in Question 1. D. (N = 59)*

Category	*Examples*	**Distribution Number (%)**
I. Right answer. Undefined value or Not a certain value.	*Not number. Undefined expression.*	**20 (34%)**
II. Right answer without reasoning.	*Not number.*	**9 (15%)**
III. Wrong answers. Without reasoning or with meaningless Reasoning. Or, right answers With erroneous reasoning.	*1. A number.* *2. A number. The subtraction of two integer numbers is a number.*	**8 (14%)**
IV$_a$. The misconception: the difference of two equal numbers is 0 (zero). ($a - a = 0$).	$\infty - \infty = 0$	**19 (32%)**
IV$_b$. The misconception: the difference of two equal Numbers is a number. ($a - a =$ a number).	*The subtraction of two numbers is a number.*	**2 (3%)**
V. No answer.		**1 (2%)**

From the above categorization (Table 5) it turns out that 34% (category I) of our PST sample give right answers. For other 15% of the sample (category II) we cannot claim the above but we cannot claim the opposite. The majority 51% (categories III, IV and V) gives wrong answers or don't answer. It turns out also that 35% (categories IV$_a$ and IV$_b$) of our sample fall with the two misconceptions $\infty - \infty = 0$ (32%) and $\infty - \infty = a$ number (3%) because the difference of two equal numbers is 0 or a number respectively. This is because ∞ is a number which is of course wrongly mode of thinking.

Question 1.E. (2^{∞})

Table 6. *Distribution for the Categories in Question1. E. (N = 59)*

Category	*Examples*	**Distribution Number (%)**
I. Right answer. Undefined operation or, not a certain value Or, $2^{\infty} \to \infty$.	*1. Not number, because it is not a certain value.* *2. Not number, because $2^{\infty} \to \infty$, the infinity is not a number*	14 (24%)

II. Right answer $2^{\infty} = \infty$ instead of $\lim_{x \to \infty} 2^{\infty} = \infty$.	*not a number.* $2^{\infty} = \infty$.	11 (19%)
III. Right answer without Explanation.	*not number.*	6 (10%)
IV. Wrong answer without explanation or, nonsensical answers.	*1. A number. A number* $2^{\infty} \to 0$. *2. A number.*	26 (44%)
V. No answer.		2 (3%)

From the above categorization (Table 6) it turns out that 43% (categories I and II) of our sample gives right answer. For other 10% (category III) of our sample we cannot claim the above but we cannot claim the opposite. The remainder 47% of our sample gives nonsensical answers or don't answer.

Right answers without reasoning is one of the interesting aspects in this study. This kind of answers triggers us to be ***suspicious***. Table 7 provides information of PST who belongs to this category in questions 1. A – 1. E, but gave wrong definition in Question 2. It turns out that most, or, all of them gave wrong definitions in Question 2.

Table 7. *Distribution for PST who gave Right Answers without reasoning in Questions 1. A – E, but gave wrong Definitions in Question 2*

Question	**1.A (N = 9)**	**1.B (N = 6)**	**1.C (N = 8)**	**1.D (N = 9)**	**1.E (N = 6)**
	∞	$0 \times \infty$	$\infty - 2^{100}$	$\infty - \infty$	2^{100}
Distribution	**7**	**5**	**7**	**8**	**6**

Reasoning, as well, is another interesting aspect in this study. From the above categorization (Tables 2-6), it turns out that PST of our sample don't have the ability of explaining their answers according to the correct theory of the infinity concept, i. e according to the formal definition of the infinity concept. Instead of that, they base their reasoning according to the concept images which are evoked in their mind. Their reasoning still based of the intuitive level of thinking. Similar results were found in other studies (Rasslan & Tall, 2002; Rasslan & Vinner, 1997, 1998; Kolar & Kadez, 2012). The gap between the use of the formal definition and ability of basing the reasoning on the formal definition is still deep also in numerical problem solving tasks connected to the infinity concept.

Discussion

One of the goals of this part of the study is to expose the concept definition of the infinity concept held by PST whose majority is mathematics. It turns out that for only 10% of our sample knew the formal definition. The fact that 90% of the PST in our sample gave erroneous responses for the infinity concept definition shocked us mathematically and pedagogically. Mathematically, because the passed successfully academic courses of pre-calculus, calculus and advanced calculus, set theory and pedagogically, because they are going to be mathematics teachers.

Second goal of this part of the study was to expose some common images of the infinity concept held by PST. This has a direct implication for teaching and learning such concept. The *images of the infinity concept* exposed in this study are: ***very large number, not a certain value, not a number, the largest number founded in R, infinity of numbers or infinity of integer numbers, undefined expression, the whole numbers.*** Some of these images founded in other studies (Kolar & Cadez, 2012). We think that some of such images like: ***very large number***, ***not a certain value, undefined expression*** are deeply internalized in the mind of our PST from other lower stages of studying mathematics. When a student in elementary, middle school requested for instance; to calculate numbers in dividing by zero tasks $(\frac{a}{0},\ \mathrm{a} \neq 0)$, or requested to solve $ax = b,\ (\mathrm{a} = 0 \text{ and } \mathrm{b} \neq 0)$ in equations solving tasks, argumentations which including such images are acceptable by most of the teachers because the ***intuitive*** approach needed in this stage of the development of such a concept like ***infinity***. Basing of the correct theory, the formal definition of the concept comes later, so we believe, in high stages of the learner. This occurs when the learner studying academic courses like set theory or advanced calculus. Here is the big fault, in our PST sample it was ***not occurs.***

The exposure of misconceptions like $0 \cdot a = 0 \text{ when a is } \infty$ or, $a - a = 0 \text{ when a is } \infty$ was the third goal of this part of the study. The results show that 71% and 7% (respectively) of our PST sample have these misconceptions. This result did not mention in other studies (Kolar & Cadez, 2012). This result hit us hard because of its dimensions. 71% of the PST in our sample thought that"∞ is a number" or because the thought that "a multiple of any numbers by zero is zero". We do not claim that the other 7% of our sample who failed in the other misconception ($\infty - \infty = 0$) are negligible. These dimensions are not expected.

Taking into account the huge difficulties (including the misconceptions) mentioned in this study and also in (Kolar & Cadez, 2012; Falk, 2010; Tall and Tirosh, 2001), at least, some doubts should be raised if the given approach to the infinity is the most effective way for teaching such a concept for PST.

The struggle between the *correct mathematics theory* and the *intuitive reasoning* is the *main factor causes such difficulties*, mentioned in this study. We believe that a systematic mathematical training according to the *correct mathematics theory* of infinity would contribute greatly to a better understanding of problems on infinity among PST.

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