Genesis of Routines: Mathematical Discourses on the Equal Sign and Variable through Commognitive Perspective

The purpose of this study is to explore the mathematical discourses of high school students and their mathematics teacher on the equal sign and variable in the classroom context. The participants of this case study are ninth grade high school students and their mathematics teacher. The data was analysed in terms of participants’ routines, specifically ritual and explorative routines in the classroom context through commognitive perspective. Results indicate that teacher discourse has governed process-based routines that have roots in elementary grades. Students have more focus on operational approaches instead of algebraic reasoning while solving equations, and this finding conveys challenges on the construct of variable.

Keywords: equal sign; variable; commognitive perspective; routines; high school

Introduction

Equal sign and variable have substantial role in the international and national K-12 curriculums and mathematics education research (MoNE, 2018; NCTM, 2005). Students have been exposed to the equal sign and variable since the early grades (primary school) in most mathematical topics (e.g., operations, ratio, first-order equations and functions) throughout the later grades. The equal sign and variable are the fundamental components of algebraic thinking and reasoning (Kieran, 2004). Algebraic thinking and reasoning enable students to consider mathematical concepts in an objectified manner instead of an arithmetic operation that needs to be solved (Kieran, 2004). Kieran (2004) recommends to focus on algebraic thinking, not just the calculation of an operation, on numbers and variables not just numbers; comparing algebraic expressions for equivalence based rather than on numerical calculation; and the meaning of equal sign, to develop students’ algebraic thinking. So, it is fundamental to understand the meaning of signs and its relationship with mathematical objects. Sign can be a mathematical symbol, a mathematical statement, a mathematical expression and a mathematical object in the context of mathematics education (Berger, 2004). Tachieli & Tabach (2012) expressed that mathematical objects exist in between symbols rather than in any of them, so none of the mathematical objects can be defined through a concrete object. Students can be engaged with the mathematical object and communicate with other participants of the discursive community for developing mathematical ideas using mathematical sign (Berger, 2004).
Studies have focused on elementary mathematics teachers’ knowledge of student thinking on core algebraic concepts, such as the equal sign and variable (Asquit, Stephens, Knuth, & Alibali, 2007). Some studies also focus on the challenge transition from operational view to algebraic thinking (Asquit et al., 2007; Bednarz, Kieran, & Lee, 1996; Kieran, 1992; Wagner & Kieran, 1989). Kilpatrick, Swafford and Findell, (2001) observed that students consider the equal sign a left-to-right directional signal. Similarly, researchers have identified that algebra in elementary grades is mostly based on equation manipulation, instead of algebraic reasoning (Asquit et al., 2007; Carpenter & Levi, 2000; Kieran, 2004; Schifter, 1999). Steinberg, Sleeman and Ktorza (1990) concluded that most students do not interpret the meaning of two given equations that are equivalent; instead, they focus on solving equations as an operation. Researchers argued that treating the equal sign as an operational manner can convey challenges in algebraic thinking and mistakes in solving equations with missing numbers (McNeil & Alibali, 2005; Powell, 2012).

Researchers have mentioned that students should interpret the equal sign as a relational view that considers the symbol the same rather than an operational view that considers the symbol as an arithmetic operation (Falkner, Levi, & Carpenter, 1999; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kieran, 2004; Rittle-Johnson & Alibali, 1999; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Sherman & Bisanz, 2009).

One of the struggles of students while dealing with equations that include unknowns is the concept of variable. Variables are an integral part of equations especially starting from elementary grades and also have a substantial role in mathematics education research era (Küchemann, 1978; MacGregor & Stacey, 1997; Philipp, 1992; Usiskin, 1988). The results of these studies indicate that students encounter challenges on the use of literal symbols in algebra. Küchemann (1978) explored that students (13-, 14- and 15-year-olds) recognise literal symbols as objects, whereas few of them interpret such symbols as unknown numbers with a fixed value as specific unknowns, generalised number that represents multiple values, or variables that represent a range of numbers. Moreover, interpreting letters as variables involves changes in values in a systematic manner (Küchemann, 1978). In sum, developing an understanding of equivalence and variable is essential to algebra and the ability to use them and algebraic thinking and reasoning (Knuth et al., 2005).

By extensively considering the literature, researchers have separately explored the understanding of students and their teachers on equation and variable through cognitive perspectives mostly at the elementary grades. Kieran (2004) suggests that a wider analysis than the existing literature is needed to have rigorous and comprehensive findings. Moreover, Sfard (2016) mentioned that more research on the relationship between teachers’ teaching and their students’ learning is needed. However, to date, little research has focused on the discursive practices of teachers and students on the concept of variables and the equal sign at the high school level. The purpose of this study is to explore the mathematical discourses of high school students and their mathematics
teacher on the equal sign and variable in the classroom context. Thus, our research question is, ‘What are the mathematical discourses of high school students and their mathematics teacher on the equal sign and variable in the classroom context?’

Theoretical Framework

I utilised the commognitive perspective in this study. Commognition is a hybrid word that is combination of communication and cognition (Sfard, 2007). The commognitive perspective formulates thinking as self-communication, and this formulation eliminates the dichotomy between thinking and communication (Sfard, 2008). Thinking is the activity of communication with oneself (Sfard, 2012). Mathematics can be seen as a discourse which is a special type of communication (Sfard, 2008). In order to be part of the same discourse community, participants do not have to actually communicate; however, participation in the communicational activities enrich participants’ sense of belonging to the wider community of discourse (Sfard, 2007). According to commognition, mathematical discourses can be identified through word use that refers to mathematical key words; visual mediators that refers to graphs, diagrams, algebraic notations and figures; routines that are repetitive patterns; and endorsed narratives that are substantiated by other elements of discourses (Sfard, 2008). Endorsed narrative is “regarded as reflecting the state of affairs in the world and labelled as true” (Sfard, 2008, pp.298). For instance, endorsed narratives are theorems, definitions and lemmas in mathematics (Sfard, 2008).

Routines are defined as a set of meta-rules that explain a repetitive action (Sfard, 2008); and a routine is a known pattern of action in a task situation (Lavie, Steiner, & Sfard, 2018). There are two different types of routines: rituals and explorations. In this study, I focus on the examination of routines, with respect to rituals and explorations. “Rituals, as sequences of discursive actions whose primary goal (closing conditions) is neither the production of an endorsed narrative nor a change in objects but creating and sustaining a bond with other people” (Sfard, 2008, pp.241). In a ritual, participants align with other members of the routines in the community, which can be considered as social approval (Berger, 2013).

“Exploration routine whose goal (closing condition) is production of endorsed narrative” (Sfard, 2008, pp.298). “Exploration is a routine whose performance counts as completed when an endorsable narrative is produced or substantiated which contributes to a mathematical theory” (Sfard, 2008, pp.224). For instance, explorations are routines of solving equations, of proving a mathematical result or of generating and investigating a mathematical conjecture (Berger, 2013). Routines do not only involve procedures (the course of action); they also involve tasks, so analysing the patterns in a task situation is essential (Lavie et al., 2018). Closure of the routines expresses circumstances...
as a successful completion of a performance, shortly includes how routines ended in a task situation (Sfard, 2008). Explorations and rituals differ from each other mainly by the types of tasks and their closure (Sfard, 2008). Students’ previous experience precedents heavily influence the routines they use in a new task situation and learning occurs through the process of routinization of students’ actions (Lavie et al., 2018). Sfard (2008) addressed that ritual is inevitable stage in routine development, and new mathematical routines which are rituals may evolve to explorations (Sfard, 2008). Participant who does not have a clear idea of when a routine can be implemented and may eventually be capable of implementing it independently (Sfard, 2008).

Recently, researchers using the commognitive framework have been focusing on two types of discursive routines: rituals, which are process oriented, and explorations, which are outcome oriented (Lavie et al., 2018). One of the strengths of the commognitive perspective is to analyse the actions and utterances of the participants of a mathematical community, which allows the exploration of communication failures (Authors, 2018). According to mathematical authorities, one of the reasons of the mathematical inconsistency of the discourses of the participants may be due to the communication of the participants with other sources (participants or textbook/tasks) in the mathematical community and their previous experiences. Hence, if mathematically inconsistent concepts are found in the communication, which are labelled as communication failures, the reasons of communicational failures should be explored to comprehend the discourse of the community in general and student discourse in particular.

Methodology

Context of the Study

Constructivist perspectives have governed the mathematics curriculum in Turkey (Zembat, 2010); however, the main teaching approach in Turkey is still direct instruction (Authors, 2018). Learning outcomes on equation and variable start at the sixth grade in the middle school in Turkey’s education curriculum. At this grade level, students are expected to comprehend algebraic expressions. The letters in algebraic expressions represent the numbers and defined as variable (MoNE, 2018). Algebraic expressions, equity and equation are learning domains at the seventh grade. Students are supposed to comprehend concept of equity, subtract and add algebraic expressions and solve first-order one-variable equations. At the eighth grade, algebra has a larger part in the curriculum that is comprised of algebraic expressions, linear equations and inequalities. Students are expected to realise algebraic expressions and factor algebraic expressions. The foundations of algebra are laid in the sixth and seventh grades. Fundamental concepts, such as equation and variable, have a
wider place in the middle grade. Then, at the ninth grade, students are expected to solve first-order one-variable equations.

Participants and Data Analysis

The participants of this case study are ninth grade (15-year-olds) high school students and their teacher Mrs. Seda (a pseudonym). I selected Mrs. Seda’s classroom as the participant for our study because she is willing to participate in the study and she communicated their experiences in an expressive and reflective way as a form of purposeful sampling for a rich and in-depth data collection (Patton, 2002). The data for the study were collected through two classroom observations, each lasting for 45 minutes. I focus on the utterances and actions of the students and teacher while transcribing the classroom observations. Analysing the utterances and actions of the participants enable us to investigate communicational failures in the classroom discourse (Authors, 2018). The classroom observations were conducted in the participants’ native language and then translated from Turkish into English. The transcripts of the classroom observations included participants’ utterances and actions. The data were analysed in terms of participants’ routines, specifically ritual and explorative routines in the classroom context (Sfard, 2008).

Results

In this study, I focused on the discourses of the teacher and students on variable and equation in the classroom context at the high school level. During the two classroom observations, the teacher and students worked on 16 tasks and the students clarified the questions on the tasks and on the topic of first-order one-variable equations. I analysed the routines of the participants in accordance with the tasks, routine procedures and closures (Sfard, 2008). Thus, the tasks have a fundamental role in the genesis of the routines. I investigated the routines based on the tasks that the teacher implemented and the students clarifying questions regarding their challenges in the classroom. I observed five routines during the classroom observations on the equation and variable within the tasks that teacher implemented in the classroom (Table 1). All of the routines have an operational structure characteristic that I explored in the classroom discourse, based on the processes that can be labelled as rituals.
Table 1. Type of routines identified from classroom observations

<table>
<thead>
<tr>
<th>Routines</th>
<th>Tasks</th>
</tr>
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<tbody>
<tr>
<td>Ritual 1: Left-to-right directional signal</td>
<td>TASK1, TASK6, TASK7 (closure: not clear for students), TASK8, TASK9, TASK10, TASK15</td>
</tr>
<tr>
<td>Ritual 2: Cross-multiplying</td>
<td>TASK6, TASK7, TASK8, TASK9, TASK10, TASK11, TASK13</td>
</tr>
<tr>
<td>Ritual 3: Cancelling factors (e.g. cancel x cubed)</td>
<td>TASK 2 (closure: not clear for students), TASK3 (closure: not clear for students), TASK4, TASK5, TASK6, TASK7, TASK8, TASK9, TASK10, TASK11, TASK12, TASK13, TASK14, TASK15, TASK16</td>
</tr>
<tr>
<td>Ritual 4: Deciding solution by an equation (If (-15 = 5), then solution set is empty; if (0 = 0), then the solution set is composed of infinite numbers)</td>
<td>TASK5 (closure: not clear for students), TASK14 (closure: not clear for students), TASK15, TASK16</td>
</tr>
<tr>
<td>Ritual 5: Going backwards with balloons</td>
<td>TASK10 (closure: not clear for students), TASK11 (closure: not clear for students)</td>
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</table>

During the classroom observations, the teacher discourse was governed with two rituals (Rituals 1 and 2), which are left-to-right directional signal and cross-multiplying. These routines have the characteristic of the operational structure, based on the processes. The following is an excerpt from the classroom discourse of the left-to-right directional signal and cross-multiplying.

1. Teacher: Find the solution of this equation [\(\frac{2}{x-1} - \frac{3}{1-x} = 10\)]. What should we do?
2. Student1: Can we do cross-multiply?
3. Student2: By multiplying.
4. Teacher: There is no multiplication here. Yes, you can say (pointing another student).
5. Student2: We can multiply one of them (showing one of the fractions) with a negative sign, and then we can do cross-multiplying.
6. Teacher: Multiplying one of them with a minus sign, why?
7. Student2: To equalise the denominators.
8. Teacher: So, to equalise denominators, can I write like this? [starting doing operations] What did I do this denominator? I bracket the minus sign [bracketing the minus sign to the fraction 3/(1-x)], right? 1 – x means x – 1 with the minus bracket. Multiply minus with minus, what will it be?
9. Student1: Plus
10. Student4: Is the answer 4?
11. Teacher: Now, I can add with the same denominator. Are the denominators equal? 5 over x – 1 is equal to 10.
12. Student5: Cross-multiplying.
13. Teacher: 2x – 2. We throw this here [showing the other side of the equation]. What happened? 3? 3 is equal to 2x, x is equal to 3/2.
Students have accustomed to work on the left-to-right directional signal and cross-multiplying rituals during their elementary grades. However, they have no previous experience on the other rituals (Rituals 3, 4 and 5). Thus, the struggles on Rituals 3, 4 and 5 have been revealed during the classroom observations. The genesis of the routines may have an explorative structure; however, the teacher discourse includes operational explanations on the routines based on the rules. By observing the closure of the routines, I revealed that the students’ discourse is not clear for these routines.

Cancelling ritual (Ritual 3) includes a process-based interpretation of cancelling without further exploration. I will give a classroom discourse that is initiated by a task that the teacher has presented. For instance, in Task 3, the task includes: If the given equation is a first-order equation that is dependent on x, find the solution set of an equation \[\frac{2}{x-1} - \frac{3}{x-1} = 10\].

48. Student2: Do we try to cancel \(x^3\) and \(x^2\)?
49: Teacher: Yes, cancel \(x^3\) and \(x^2\). This is third degree; this is second degree; then, what will happen to them? [showing \(x^3\) and \(x^2\)] What should it be? \(a - 2\) should be equal to zero; then, \(a = 2\) should be equal to zero; then, \(b - 3\) should be equal to zero; then, \(b\) is equal to \(3\). Then, let’s write them in the equation: \((2 - 2)x^3 + (2 - 2)x^2 + 4x + 2.2 + 4.3 = 0\) [writing the \(a\) and \(b\) values in the equation]. Here [showing \(2 - 2\)], what happened to 0; here [showing \(3 - 3\)], what happened to 0 multiplied by 0. \(4x + 4 + 12 = 0\) and \(4x + 16 = 0\) [students are also repeating the same equation].
50. Student1: Put 16 on the other side of the equation.
51. Student1: 4
52. Teacher: \(x = 4\), right?
53. Student2: \(x = -4\)
54. Student3: \(x = -4\)
56. Teacher: So, the solution set is \(-4\). Yes, where do we use the curly bracket? For the solution sets because there are the sets of points.

Ritual 4 includes deciding the solution set of the given first-order equation through the interpretation of a final equation. For instance, if \(-15 = 5\), then the
solution set is empty, or if $0 = 0$, then the solution set is infinite without discussing the mathematical thinking behind these equations. In Task 14, the teacher asked the students: ‘If the solution set of this equation \[ m(2-x) = nx + 4 \] is infinite, then what is $n$?’ The classroom discourse was initiated by this task, and the student and teacher discourses are given below.

140. Teacher: … Now, what does it mean to have infinite elements?

143. Teacher: So, we write $a$ is equal to 0 and $b$ is equal to 0 for the equation $ax + b$. If 0 is equal to 0, then what can we say?

Infinite elements, so the solution set for this equation is real numbers \[ \text{[writing this]} \], right? Okay, let’s organise this: $2m$ minus $mx$ minus $nx$ minus 4 is equal to 0. Let’s have the bracket of $x$, $-m$ minus $n$ plus $2m$ minus 4 is equal to 0 \[ \text{[writing this]} \]. Here is the expression with $x$ \[ \text{[showing the coefficient of expression with $x$]} \]. What will be the coefficient of the expression with $x$? \[ \text{[there is no answer from the students]} \]. It needs to be zero: 0 multiplied by $x$ plus 0 is equal to 0. Okay. What will this be \[ \text{[showing $2m - 4$]} \]? Here, it will be 0. So, $-m - n$ is equal to 0; from here, if I take $n$ to the other side of the equation, $m$ is equal to $-n$ \[ \text{[writing this]} \]. $2m$ minus 4 is equal to 0, and then $m$ is equal to 2, right? \[ \text{[showing $2m - 4 = 0; m = 2$]} \]. If $m$ is equal to 2, then what is $n$?

144. Teacher: $-2$.

145. Teacher: $-2$ \[ \text{[writing this]} \].

In the given classroom discourse, the teacher discourse led the student discourse on ritual 4, which has a procedural structure, which is comprised of
the following: i) if you find one type of equation \((-3 = 5)\), then the solution is empty; ii) if you find another type of the equation \((2 = 2)\), then the solution is infinite. However, the relationship between an equation and a solution set is not extensively discussed in the classroom discourse to make the discourse transparent for students. ‘Why is there a relationship between an equation and a solution set? What is a solution set? What is an empty or infinite solution set?’ All of these questions need to be clarified in the classroom discourse.

In Ritual 5, the teacher initiated a classroom discourse by giving Task 10:

Find the value that satisfies \(x\) in the equation?

34. Student1: Balloon.
35. Teacher: Balloon. Okay, you named this as a balloon.
36. Student2: We call this equation.
37. Classroom: This is a balloon.
38. Student3: We call this an unsolvable equation.
39. Teacher: In this type of tasks, we start to solve at the end [showing the denominator \(5 - 1/x - 1\)]. However, I did not solve this type of tasks like this [showing the denominator \(5 - 1/x - 1\)]. So, what is the product? [showing 2].
40. Classroom: 2.
41. Teacher: 2. Okay. What should I add to 1 to have 2? [showing 1]
42. Student1: 1
43. Student2: 1
44. Teacher: Ok, [closing 1], cancel this; what will be here? [showing the fraction \(\frac{6}{x-1}\)]
45. Student1: 1.
46. Teacher: To have 1 here, what will be the value of here [showing \(\frac{5-1}{x-1}\)]?
47. Student1: 6
48. Teacher: 6 [writing \(\frac{5-1}{x-1}\)]. 5 minus 1 over \(x - 1\) is equal to?
49. Student1: 6.
50. Teacher: 6. So, what is the value of here? \(-1\)? [encircling the minus 1 over \(x \text{ minus 1}\)]
51. Student1: Yes.
52. Student2: Yes.
53. Teacher: \(-1\) and 5, what is in the front? a minus sign [encircling 1/x – 1]. 5 minus 1 is 6 [showing the equation]. So, what is the value of 1/x-1? [writing 1/x – 1 = \(-1\)].
54. Student: \(-1\).
55. Teacher: −1. Now, by cross-multiplying, 1 is equal to negative x plus 1. We put 1 on the other side of the equation, so x is equal to 0.

If we write x is equal to 0, what will be the answer? The answer is 2.

We write 0, then what happened? Minus 1 minus 1 minus becomes plus 5, plus 1 6, 6 over 6 1, 1 plus 1, 2, so it is true [by substituting 0 for x into the equation, and controlling if the equation is satisfying or not], right? By substituting, I am controlling if I found the right answer. Thus, the solution set is [writing the board ÇK = 0 (ÇK is the abbreviation of solution set in Turkish)]. Is this challenging?

56. Student1: Yes.

57. Student2: Yes.

58. Classroom: Yes, challenging. (Most of the students in the classroom says: ‘Yes, challenging’)

58. Student3: It is not challenging; it is annoying.

In the given classroom discourse, the students encountered challenges on the task, the teacher discourse is dominant in the excerpt and the students are mostly trying to imitate the teacher discourse. One of the significant points of this classroom discourse is the closure of students’ discourses, which includes ‘this is really challenging’ and ‘annoying and long’. Such statements are indicators of the struggles that the students encountered. Moreover, the student discourse on the variable had struggles. Below is the classroom discourse about the variable.

15. Teacher: …When I say first-order equation, there needs to be one [showing the exponent of x].

is a first-order equation. How many variables are there?
16. Student1: 3
17. Student2: 2
18. Student3: 1
19. Student4: 3, a, x, b
20. Student5: No, a and b are natural numbers, so there is just one variable. The others [a, b] represent numbers.
21. Teacher: She said just one. So, here is just one variable, what should I call this? First-order one-variable equation. Okay? So, what is a and b? They are real numbers.

In the classroom discourse, students consider a, x and b as all variables because they are unknown in a given equation. The students also consider all unknowns as variables. The teacher stressed that a and b are real numbers but did not mention the definition of a variable. A variable can also be a real number that can be changed at once in a range of numbers. Thus, the teacher discourse on the variable was not clear for the students. After the implicit discussions on the variables in the classroom, I revealed a communicational failure on the difference between the letter as variable and unknowns (x, a).

110. Teacher: …If the value that satisfies x for the equation is −2, what is a? How can I ask this question in another way?
111. Teacher: If the root of the given equation is −2, what is a?
112. Student1: But it needs to be in accordance with the x variable, right?
113. Teacher: It is in accordance with the x variable, but there is no other variables in the equation.
114. Student1: There is a.
115. Student2: a is a number.
116. Student1: When the root of an equation is mentioned, it needs to say that x is a variable.
117. Student3: a is a number; a is a number!
118. Student1: Ha, okay!

In the given classroom discourse, the teacher considers ‘x’ as a variable and ‘a’ as an unknown. However, for students, differentiating between the variable and an unknown is still challenging, so they both consider ‘x’ and ‘a’ as variable. ‘What is unknown? What is variable?’ They had to be clarified for the students. At that point, the realization of a mathematical definition and statement come to the forefront.

Another challenge in the classroom discourse is the ‘root of an equation’. One of the students has initiated a question about the comprehension of the root of an equation. The classroom discourse on this topic is given below.

86. Student1: What is the meaning of the root of an equation?
87. Teacher: What is the root of an equation? What is root?
88. Student2: x
89. Student3: $x$
90. Student4: Variable
91. Teacher: So, is it $x$? If the root of an equation is $-1$?
92. Student5: Result
93. Teacher: So, what is the value of $x$? If it is equal to $x$?
94. Student6: Solution set
95. Teacher: What is the solution set? $-1$. Let’s look at the solution sets in the other tasks.
96. Student1: $a$ is 5, right?
97. Student3: $a$ is 5.
98. Teacher: Ok, listen to me. We found that $x$ is equal to $3$, ok? [showing the solution of the empty set], all of these [showing the solution $x = 3/2$]. The roof of all of these equations [showing all answers of the 3 tasks].
99. Student1: Root.
100. Teacher: So, satisfying the value of this [showing the equation]. Please take note if you do not know this. Satisfying the value means the root of an equation. What do we call the value that satisfies $x$? The root of an equation.

Students interpreted the root of an equation as a variable and unknown; thus, it represents `$x$`. `$x$` can be a variable and an unknown for them. One of the students explicated `$x$` as a solution set. Then, from this approach, the teacher interpreted the root of an equation as a value that satisfies the equation. Another challenge for the students is finding the solution set in different number systems. This result may be attributed to Ritual 4, which is a conditional process-oriented routine that includes deciding the solution set if $5 = 3$ or $2 = 2$. Below is a classroom discourse based on a task that the teacher asked: ‘Find the solution of $2x - 11 = 0$ in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$’.
59. Teacher: …What does this mean?
60. Student1: Natural number, whole number.
61. Student2: Natural number.
62. Teacher: Find the natural number, whole number, rational number and real numbers. Okay, find the solution [students are working on the task].
63. Teacher: Yes, what is the answer in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$?
64. Student1: 5 in $\mathbb{N}$, 5 in $\mathbb{Z}$, 11/2 in $\mathbb{Q}$, 11/2 in $\mathbb{R}$.
65. Teacher: So, you found 5 in $\mathbb{N}$, 5 in $\mathbb{Z}$, 11/2 in $\mathbb{Q}$ and $\mathbb{R}$; any other answers?
66. Student2: 6 in $\mathbb{N}$ and $\mathbb{Z}$, because it becomes 5, 5, when we round up.
67. Student3: I found 6.
68. Teacher: Congratulations! Any other ideas?
69. Student4: 11/2 in $\mathbb{Q}$ and $\mathbb{R}$ I could not find any solutions in $\mathbb{N}$ and $\mathbb{Z}$.
70. Student5: Can we say 5, 5 in real numbers?
71. Teacher: Yes, you can say in real numbers. Is 5, 5 not an element of the real number? Or 11/2. So, what was our question? $2x - 11 = 0$, $x = 11/2$. Whose element is 11/2?
72. Student6: Rational and real
73. Student7: Rational and real numbers
74. Teacher: So, real numbers and rational numbers. As you know, real numbers subsume rational numbers.

75. Student1: Yes.

76. Teacher: The answer in real numbers and rational numbers is 11/2. However, if I ask you the solution set in N. The solution set in N and Z is an empty set. If you say ‘It is 11/2, so I round up and take 5 or 6’, this is impossible. You cannot make up; you cannot round up! Can the answer be round up? Substitute 6 in the equation; does it satisfy? Substitute 6, 2 multiplied by 6 minus 11 is equal to 0, is this right?

77. Student1: Nope, it is equal to 1.

78. Teacher: So, what happened? I round up and take 5; round up and take 6 is not possible. What do you round up? This is impossible. So, what do we say for the solution set in N and Z?

79. Student1: Empty set

80. Teacher: Empty set. What is the solution set in rational numbers Q?

81. Student1: 11/2

82. Teacher: 11/2. The solution set in R is 11/2. Okay?

According to the classroom discourse, the students’ realization of the solution set depends on the operational view as rituals. When the students encounter different types of task that need an explorative realization, they have struggles to interpret the solution set and find their own way from their discourse. As a conclusion, the root of an equation is unknown and variable.

Discussion & Conclusions

The purpose of this study is to explore the mathematical discourses of high school students and their mathematics teacher on the equal sign and variables in the classroom context. The results indicate that the teacher and student discourses are governed with process-based routines, where some (Rituals 1 and 2) have roots in elementary grades. Aligned with the literature, I explored that students consider the equal sign as a left-to-right directional signal (Kilpatrick, Swafford, & Findell, 2001); besides, students have more focus on operational approaches instead of algebraic reasoning and mathematical objects while solving equations in this study (Asquit et al., 2007; Berger, 2004; Carpenter & Levi, 2000; Kieran, 2004). The foundations for algebra are laid in the sixth and seventh grades in Turkey’s curriculum. Fundamental concepts, such as equation and variable, have a wider place in the middle grade. Then, at the ninth-grade level, students are expected to solve first-order one-variable equations and consider equations from an operational view. However, the literature stressed that if students do not have a relational understanding of the fundamental concepts, they have struggles while solving equations (Kieran, 2004). Besides, if students do not have thorough understanding of the sign by relating it with mathematical objects, they can only perform mathematical sign
in restricted ways and may have challenges while implementing them (Berger, 2004).

Student discourses have implicit character with a fuzzy thinking on unknowns. This fuzziness on unknowns conveys struggles on basic mathematical constructs, such as variable and root of an equation. One of the reasons of the struggles on unknowns may be due to implicit teacher discourse on unknowns and variables that teacher possesses and process-based routines that do not have an explorative structure. As Berger (2004) mentioned, students’ use of signs within a social community allow them to develop meaning of the sign which is compatible with the community. In this study, students could not have change to adopt the teacher discourse due to implicit structure of the discourse on unknowns and variables. The other reason may be students’ precedent routines on the concepts of unknowns. During their elementary grades, if they consider unknowns as a tool to perform operations and do not have relational understanding and relate unknowns with mathematical objects, this may lead to challenges. Aligned with the literature, the teacher discourse on the usage of letters as specific unknown, generalised number or variable is not transparent and explicit for students, and this led up to challenges for students on differentiating between variable and unknowns in this study. Küchemann (1978) stressed that mathematics teachers have tendency to use the blanket term ‘variable’ for any and all letters in generalised arithmetic. However, it is fundamental to be aware of using letters as objects or specific unknowns or generalised numbers or variables in algebra to improve algebraic thinking (Küchemann, 1978).

Mathematical tasks need an explorative and ritual realisation. If the teacher has implemented the same type of task from the operational perspective, then the student discourse can be consistent and dominant on the operational view like in this study and students can easily imitate teachers’ discourse. Sfard (2008) expressed that imitation is an obvious way to enter a new discourse (Sfard, 2008). Rituals include imitations of other’ routines which can be seen as social approval (Berger, 2013). In this study, students might use rituals in order to get social approval from their teacher. However, when students face tasks that include explorative characteristics, then students have difficult times to interpret the teacher discourse. When the teacher discourse is not transparent for students, students may have communicational failures (Authors, 2018). The characteristic of the tasks is one of the important components of routines’ genesis. Aligned with the results of Nachieli & Tabach (2012), I explored a strong relation between tasks and routines. In addition, rituals are highly associated with tasks, which are very restricting (Sfard, 2008). In this study, students are accustomed to work on the left-to-right directional signal and cross-multiplying rituals during their elementary grades. However, they have no experience on the other rituals from their precedents. Thus, I observed their struggles on Rituals 3, 4 and 5 in the classroom discourse. The reason for this may be that the routines’ genesis has an explorative structure that emerged from the tasks. However, the teacher discourse includes operational
explanations on the routines based on the rules. This led up to challenges for students and unclear student closure of the routines. Hereby, different rituals cannot be seen as interchangeable since they produce the same closures (Sfard 2008).

As a conclusion, to have more explicit and transparent discourses for students, different types of routines should be used in the classroom. To diversify routine types in the classroom, the characteristics of tasks have a substantial role. Teachers should implement operational and explorative tasks in the classroom. These types of tasks should convey the operational and explorative type of routines. Moreover, it is a necessity to conduct longitudinal researches that explore extensively at which level, when and how to implement operational and explorative type of routines. Mathematics educators should focus on different performances in a wider discursive context to analyse one’s routine in detail, since the difference between ritual and exploration lies on when of them conducted (Sfard, 2008).

References

Authors, (2018).


Sfard, A. (2017). Ritual for ritual, exploration for exploration: Or, what learners are offered is what you get from them in return. In J. Adler & A. Sfard (Eds.), Research for Educational Change: Transforming Researchers’ Insights into
