General Solution of Two-dimensional Projectile Motion with Air Resistance

In this study, two-dimensional projectile motion is considered under the effect of a general power law model of air resistance. Classically, a projectile is treated as a point mass with mass m moving in a uniform gravitational field. The projectile is launched from the ground with an angle α to horizon. the drag force is assumed to be proportional to the speed raised to the power n. The analysis of the problem is performed using Cartesian coordinates. A general exact parametrical solution (with respect to the angle of motion) is derived for any power n, following simple steps: 1) find the speed in the direction of the axis x (horizontal – no gravity); 2) find the vertical component of the speed; 3) find the time; 4) find the horizontal position of the projectile; and finally, 5) find the vertical position of the projectile. Steps 1) and 2) give explicit closed form equations and the rest are given by exact integrals which can be solved numerically. In this study spreadsheet calculation are performed using trapezoidal rule of integration. The cases of motion in a vacuum and linear drag law are used to check the accuracy of the numerical calculations. The importance of the proposed study is three-fold: a) The method of the derived solution is new, and couldn't be found elsewhere; b) The derived equations make it possible to use spreadsheets for presenting the subject (no programming ~~capabilities~~ is required)~~,~~ and thus, serve as a tool to enhance teaching; c) The derived equations are general for any power n; thus, the same procedure could be used to find the position of the projectile at any time.

Keywords: *projectile motion, air resistance, general power law*

Introduction

Motion of bodies in two dimensions with constant vertical acceleration and zero horizontal acceleration are called projectile motion [1]. Throwing a ball off a tower, firing a cannon, driving a golf ball off of a tee, or shooting a basketball- these are just a few examples of projectile~~'s~~ motion [2].

Projectile motion has a long history. Aristotle's theory of projectile motion was based on everyday life observation, for instance, if an object is moving, then, something must be moving it [3]. When guns were developed in the 14th century, a more accurate theory was necessary to describe a projectile's motion. It was not until Galileo and Newton worked on the problem that a better theory of ideal projectile motion was reached. [3].

According to [4], Galileo introduced inertia and the theory of projectile~~'s~~ motion, and thus he was the first in history to solve a projectile's motion. He solved the problem for the ideal case of no air resistance. Furthermore, it was known that projectile motion was a special case of Newton's second law. The problem of a projectile's motion with air resistance was considered by Bernoulli [5, 6]. For the linear case where the drag coefficient is proportional to the projectile's speed, there exists an analytic solution [7, 8]. For the general case where the drag coefficient is proportional to any power of the speed, there is no closed form solution. The quadratic drag case was studied extensively using different approaches including numerical calculations; analytic approximations; simulations, and by introducing exact integrals. [5 – 7, 9 - 13]. The projectile's motion with a general power law of air resistance was studied by using path coordinates (a projectile's speed and angle of motion) [13]. The projectile's position was presented parametrically using exact integrals.

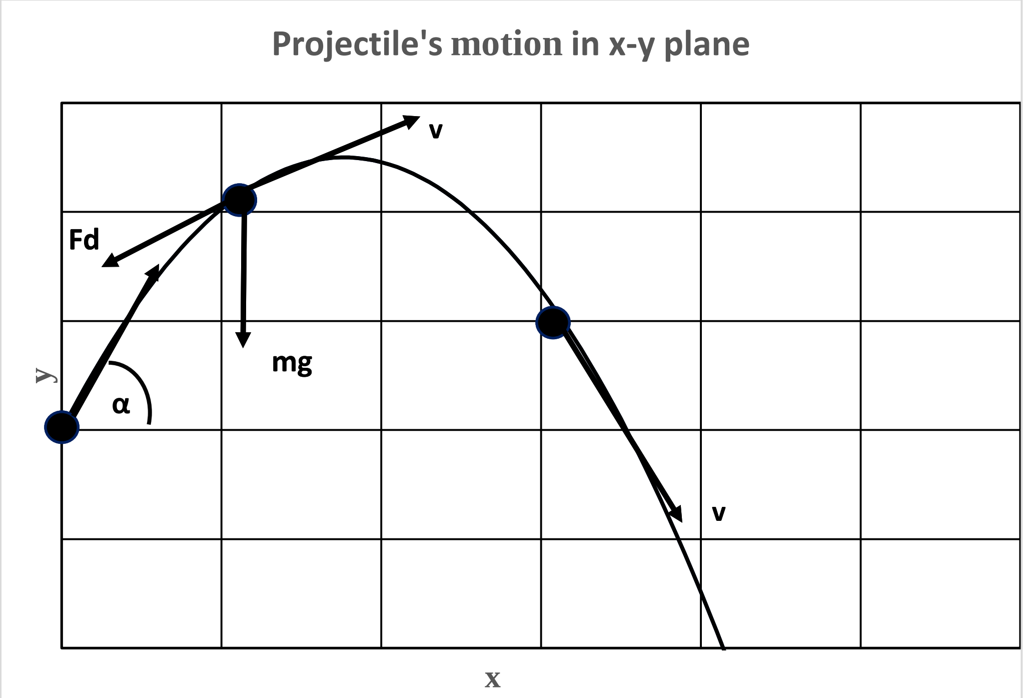
In this study, the two-dimensional projectile's motion with a general power law of air resistance model is reconsidered by using cartesian coordinates. The equations of motion were derived from Newton's second law. The horizontal velocity component and the angle of motion are used as the independent variables. This choice of variables made it possible to present the time and position of the moving body by exact integrals which are suitable for spreadsheet calculations. The benefits of this analysis are threefold: a new method of solution is presented; a general model is suggested to study a projectile's motion with a general power law of air resistance; and spreadsheet calculations are demonstrated as a pedagogical tool.

The rest of the manuscript is arranged as follows: the problem statement and the derivation of the equations of motion are given in section 2, the general procedure to solve the equations of motion and special cases are considered in section 3, spreadsheet numerical examples are given in section 4, and finally, summary and conclusions are given in section 5.

Two-dimensional Projectile motion with air resistance

Two-dimensional motion of a projectile experiencing a constant gravitational force and an air drag force which is proportional to the *n* power of the projectile's speed, is considered. The projectile's motion is described by means of cartesian coordinates. Without loss of generality, the projectile's motion starts from the origin with initial velocity v_0 and directed with angle α above the horizontal (see figure 1).

Figure 1. *Schematics of a projectile's motion under the effects of a constant gravitational force and an air resistance force which is proportional to the projectile's speed raised to the power n*



Newton's second law for the projectile's motion is written as follows:

$$m\boldsymbol{a} = -\,m\boldsymbol{g} - \boldsymbol{F}_D = -mg - D\boldsymbol{v}^n = -mg - Dv^n\frac{\vec{v}}{v} = -mg - Dv^{n-1}\,\vec{v} \qquad (1)$$

Where $\vec{v}$ is the velocity vector and is given by:

$$\boldsymbol{v} = \vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \qquad (2)$$

And the projectile's speed v is given by:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} \qquad (3)$$

After dividing by the mass *m*, equation (1) is rewritten for *x* components of the acceleration as follows:

$$\frac{dv_x}{dt} = -Av^{n-1}v_x \qquad (4)$$

Where A is the proportionality constant *D* divided by the mass *m* and is given by:

$$A = \frac{D}{m} \qquad (5)$$

Similarly, the y component of equation (1) for *y* component of the acceleration, and is given by:

$$\frac{dv_y}{dt} = -g - Av^{n-1}v_y \qquad (6)$$

For convenience, nondimensional variables are used. The subscript *s* denotes dimensional scales as specified in table1.

Table 1. *Dimensionality factors and variables*

Variable #	variable	Dimensional factor	Non-dimensional variable
1	$v = v_0\, v^*$	v_0	v^*
2	$v_x = v_0\, v_x^*$	v_0	v_x^*
3	$v_y = v_0\, v_y^*$	v_0	v_y^*
4	$t = t_s t^*$	$t_s = \frac{2v_0}{g}$	t^*
5	$x = x_s x^*$	$x_s = \frac{2v_0^2}{g}$	x^*
6	$y = y_s y^*$	$y_s = \frac{2v_0^2}{g}$	y^*
7	$A = A_s A^*$	$A_s = \frac{g}{2\, v_0^n}$	A^*
8	$v_{x0} = v_0 \cos(\alpha)$	v_0	$v_x^* = \cos(\alpha)$
9	$v_{y0} = v_0 \sin(\alpha)$	v_0	$v_y^* = \sin(\alpha)$

Following the definitions that are given in table 1, equation (4) is rewritten in dimensionless form and is given by:

$$\frac{dv_x^*}{dt^*} = -A^* v^{*(n-1)} v_x^* \qquad (7)$$

Similarly, equation (6) is rewritten in dimensionless form and is given by:

$$\frac{dv_y^*}{dt^*} = -2 - A^* v^{*(n-1)} v_y^* \qquad (8)$$

In order to simplify the solution method, equation (7) is divided by v_x^* and rewritten as follows:

$$\frac{d\ln(v_x^*)}{dt^*} = -A^* v^{*(n-1)} \qquad (9)$$

The same simplification is used as before such that equation (8) is divided by v_y^* and rewritten as follows:

$$\frac{d\ln(v_y^*)}{dt^*} = -\frac{2}{v_y^*} - A^* v^{*(n-1)} \qquad (10)$$

By subtracting equation (10) from equation (9) and after proper mathematical manipulation, the following equation is derived:

$$\frac{d\ln\left(\frac{v_x^*}{v_y^*}\right)}{dt^*} = \frac{2}{v_y^*} \tag{11}$$

Notice that the ratio between the speed components is given by:

$$\frac{v_x^*}{v_y^*} = \frac{1}{\tan(\theta)} \tag{12}$$

Where θ is the angle of motion. By using the chain rule of differentiation ($\frac{d(.)}{dt^*} = \frac{d(.)}{d\theta}\frac{d\theta}{dt^*}$), equation (11) is rewritten respectively and is given by:

$$\frac{d\theta}{dt^*} = -\frac{2\cos(\theta)}{v^*} \tag{13}$$

Similarly, by using the chain rule of differentiation ($\frac{d(.)}{dt^*} = \frac{d(.)}{d\theta}\frac{d\theta}{dt^*}$), and by using equation (13), equation (7) is rewritten after proper mathematical manipulations and is given by:

$$\frac{du}{d\theta} = -\frac{A^*}{2}\, v^{*(n-1)} \tag{14}$$

Where $u = v_x^* = \frac{dx^*}{dt^*}$ and $v^* = \frac{u}{\cos(\theta)}$.

Equation (14) is rewritten in terms of *u* and θ and is given by:

$$\frac{du}{d\theta} = -\frac{A^*}{2}\frac{u^{n+1}}{(\cos(\theta))^{n+1}} \tag{15}$$

by repeating the same arguments, equation (13) is rewritten in terms if u and θ and is given by:

$$\frac{d\theta}{dt^*} = -\frac{2\,(\cos(\theta))^2}{u} \tag{16}$$

It is important to note that by solving equations (15) and (16) the speed in the horizontal direction and time are derived as a function of the angle θ. Then the vertical component of the speed is found by using equation (12). Finally, by knowing the time and the speed, the position of the projectile is calculated by integrating the velocity components with respect to time.

The results derived in this section are used as building blocks for the general model of two-dimensional projectile motion.

A general model of a projectile's motion in two dimensions

In this section the procedure to find the position of the projectile versus time is outlined in subsection 3.1.

General procedure

1) Solve equation (15) for u. The solution can be found by the method of separation of variables. The differential equation for u is given by:
$$\frac{du}{u^{n+1}} = -\frac{A^*}{2}\frac{d\theta}{(\cos(\theta))^{n+1}} \quad (17)$$
The general solution for u as a function of θ is given by:
$$\frac{u^{-n}}{-n} = -\frac{A^*}{2}\left(\frac{(\sec(\theta))^{n-1}\tan(\theta)}{n} + \frac{n-1}{n}\int(\sec(\theta))^{n-1}\,d\theta\right) + const (18)$$
2) Solve equation (16) for time by the method of separation of variables. The non-dimensional time is given by:
$$t^* = -\int_{\alpha}^{\theta}\frac{u}{2\,(\cos(\theta))^2}\,d\theta \quad (19)$$
3) Solve for x^* by using the definition of u. The non-dimensional horizontal position is given by:
$$x^* = \int_0^{t^*} u\,dt^* \quad (20)$$
and finally,
4) Solve for y^* by using u, equation (12) and the definition of vertical component of the speed. The non-dimensional vertical position is given by: $y^* = \int_0^{t^*} u\tan(\theta)\,dt^*$ 1)

Special cases

In this subsection, several cases are considered including motion in a vacuum and motion under air resistance with *n*=1, and 2. The first two cases are used to check the numerical accuracy. Furthermore, the last case could be solved semi-numerically and fully numerically by means of Microsoft excel spreadsheet. These calculations enable estimating the accuracy of the calculations.

Case $A^* = 0$, no air resistance

Analytic calculation

From equation (7) it is deduced that the horizontal speed component is constant (the classical result) and is given by:

$$v_x^* = \cos(\alpha) \quad (22)$$

By substituting equation (22) in equation (20), it is shown that the non-dimensional horizontal position is given by:
$$x^* = \cos(\alpha)\,t^* \quad (23)$$

Similarly, the vertical speed component is calculated from equation (8). The non-dimensional vertical speed is given by:

$$v_y^* = \sin(\alpha) - 2\,t^* \qquad (24)$$

Then, the non-dimensional vertical position is derived by substituting equation (24) in equation (21), and completing the integration, thus, the $y*$ is given by:

$$y^* = \sin(\alpha)\,t^* - t^{*2} \qquad (25)$$

Calculations based on the equations of the general model

By substituting $A^* = 0$ in equation (18), the non-dimensional horizontal speed is shown to be a constant and is given by:

$$u = \cos(\alpha) \qquad (26)$$

Then, by means of equation (12), the vertical speed is given by:

$$v_y^* = \cos(\alpha)\tan(\theta) \qquad (27)$$

The parametric relation between the nondimensional time and the angle θ is derived by means of equation (19) and is given by:

$$t^* = \frac{\cos(\alpha)}{2}(\tan(\alpha) - \tan(\theta)) \qquad (28)$$

By eliminating $\tan(\theta)$, equation (28) is rewritten in the form:

$$\tan(\theta) = \tan(\alpha) - \frac{2\,t^*}{\cos(\alpha)} \qquad (29)$$

after substituting equations (26) and (29) in equation (27), equation (24) is retrieved and thus, the projectile's position is given by equations (23) and (25) for the horizontal and vertical coordinates respectively.

Finding the position by spreadsheets

By following the general procedure, the horizontal speed is calculated from step 1, then the time and projectile's position are calculated by using Microsoft excel spreadsheet. The analytic formulas are used to check the accuracy of the calculations. It is shown that the accuracy depends on the step size of integration and on the quadrature used. As was stated before, the trapezoidal quadrature rule is used in this study. Visual demonstrations of the calculations are given in the next section.

Case $A^* \neq 0, n = 1$

Analytic calculation

In this case, the horizontal acceleration after substituting *n* = 1 in equation (7), is given by:

$$\frac{dv_x^*}{dt^*} = -A^* v_x^* \qquad (30)$$

Similarly, the vertical acceleration after substituting *n* = 1 in equation (8), is given by:

$$\frac{dv_y^*}{dt^*} = -2 - A^* v_y^* \qquad (31)$$

By integrating equation (30) and using the appropriate initial condition (entry 8 in table 1), the horizontal speed is given by:

$$v_x^* = \cos(\alpha)\, e^{-A^* t^*} \qquad (32)$$

Similarly, by integrating equation (31) and using the appropriate initial condition (entry 9 in table 1), the vertical speed is given by:

$$v_y^* = \sin(\alpha)\, e^{-A^* t^*} - \frac{2}{A^*}\left(1 - e^{-A^* t^*}\right) \qquad (33)$$

Now, starting at the origin, the horizontal position of the projectile is derived by integrating equation (32) and is given by:

$$x^* = \cos(\alpha) \frac{1}{A^*}\left(1 - e^{-A^* t^*}\right) \qquad (34)$$

By repeating the same procedure, the vertical position of the projectile is derived by integrating equation (33) and is given by:

$$y^* = \left(\frac{\sin(\alpha)}{A^*} + \frac{2}{A^{*2}}\right)\left(1 - e^{-A^* t^*}\right) - \frac{2}{A^*}\, t^* \qquad (35)$$

Calculations based on the equations of the general model

By applying the procedure of the general model for the case *n* = 1, the horizontal speed is found from equation (18) and is given by (step 1):

$$\mathrm{u} = \frac{1}{\frac{1}{\cos(\alpha)} + \frac{A^*}{2}(\tan(\alpha) - \tan(\theta))} \qquad (36)$$

Based on step 2, the non-dimensional time is calculated from the following integral (after substituting the expression for *u*):

$$t^* = -\int_{\alpha}^{\theta} \frac{1}{2\,(\cos(\theta)^2} \frac{1}{\frac{1}{\cos(\alpha)} + \frac{A^*}{2}(\tan(\alpha) - \tan(\theta))}\, d\theta \qquad (37)$$

By completing the integration, the non-dimensional time is given by:

$$t^* = \frac{1}{A^*} ln\left(\frac{\frac{1}{\cos(\alpha)}+\frac{A^*}{2}(\tan(\alpha)-\tan(\theta))}{\frac{1}{\cos(\alpha)}}\right) \quad (38)$$

In fact, after substituting equation (36) in equation (38), the expression for the horizontal speed (see equation (32)) is retrieved.

The vertical speed component is obtained by dividing equation (33) by equation (12). The output result is a relation between the vertical speed component and the angle of motion. In order to write the relation of the vertical speed as a function of time, equation (38) is rearranged such that tan(θ) is given by:

$$\tan(\theta) = \tan(\alpha) + \frac{2}{A^*\cos(\alpha)}\left(1 - e^{A^*t^*}\right) \quad (39)$$

At this point, by using equations (12), (36) and (39), and by performing proper mathematical manipulations, the expression for the vertical speed is shown to be given by equation (33).

To complete the calculations, the projectile's position is found exactly as was done in the previous subsection and the position is specified by equations (34) and (35).

<u>Finding the position by spreadsheet calculations</u>

As was done in section 3.2.1.3 and by following the general procedure, the horizontal speed is calculated from step 1, then the time and projectile's position are calculated by using Microsoft excel spreadsheet. The analytic formulas are used to check the accuracy of the calculations. Visual demonstrations are given in the next section.

Case $A^* \neq 0, n = 2$

<u>Semi-numerical calculation</u>

For cases of $n \geq 2$ the general procedure to calculate the parameters of the projectile's motion is followed as was described previously (steps 1-4). The semi-numerical calculation is made of two parts: analytic (step 1), such that the velocity components are calculated based on analytic formulas and; a numeric part (steps 2-4) which is based on the analytic results.

The horizontal speed for the case of n = 2 is found from equation (18) and is given by:

$$\frac{1}{u^2} = \frac{1}{\cos(\alpha)^2} + \frac{A^*}{2}\left(\tan(\alpha)\sec(\alpha) - \tan(\theta)\sec(\theta) + ln\left(\left|\frac{\tan(\alpha)+\sec(\alpha)}{\tan(\theta)+\sec(\theta)}\right|\right)\right) \quad (40)$$

The vertical component of the velocity vector is calculated by using equation (12), and the numerical integration is performed by means of the trapezoidal rule method.

<u>Numerical calculation</u>

The calculation of the previous section is repeated, but this time the right-hand side of equation (17) is performed numerically as before (by means of the trapezoidal rule). This calculation enables comparison between the result achieved in both subsections for n = 2. Numerical examples are given in the next section.

For further checking, the numerical calculations could be performed by using Runge-Kutta 4^{th} order method (RK4) for solving ordinary differential equations (odes). An implementation of RK4 to projectile's motion is given in Appendix 1, in which there are more details which are given for the convenience of the reader.

Spreadsheet calculations

In this section, the aforementioned general procedure is implemented in Microsoft spreadsheet to calculate the time and space coordinates both analytically when is it possible and numerically (just estimating the integrals by trapezoidal quadrature rule). The relative error (exact value – approximate value)/exact value*100% is reported for two cases, motion in vacuum (no air resistance) and for motion in air (with air resistance with power n = 1).

Arbitrarily, the range of the angle is started from the initial value of $\frac{\pi}{4}$ until -1 radian. In all calculation the initial angle is 45 degrees and the projectile is launched from the origin. The air resistance coefficient (drag coefficient) depends on the projectile's speed and properties (shape, size, cross sectional area) and air density. In this study, a baseball example with 0.145 kg mass and diameter of 0.075 m [14] were considered for which values of A^* fall in the range 0.1 – 3.0 (see appendix 2 for more details).

These calculations and comparisons are necessary and essential steps towards trusting the calculations, especially for cases where no closed form formulas are found.

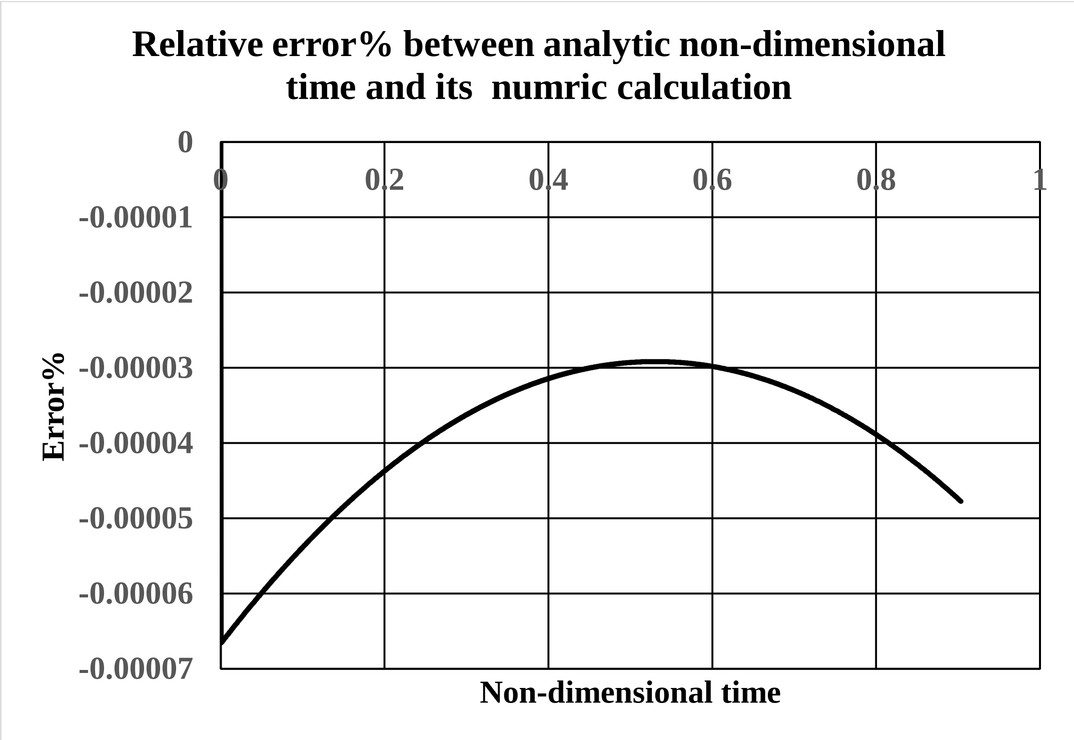
Motion in a vacuum

For the ideal case, projectile's motion in vacuum or without air resistance, the time and position are calculated analytically and numerically. The relative error% is calculated and presented in figures 2-4. The accuracy depends on the step size of the numeric calculations. In the current study, step size of 0.001 radian was used for the angle.

Figures 2-4 address the relative error in % appropriately: in calculating time (figure 2); in calculating horizontal position (figure 3); and in calculating vertical position (figure 4).

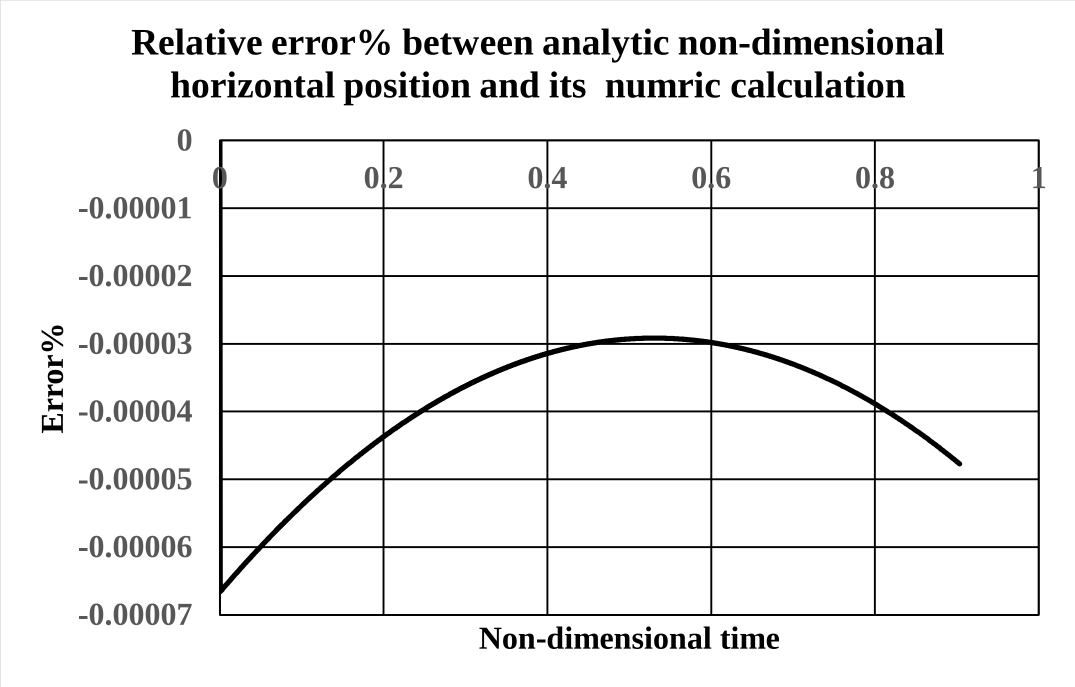
Figure 4 shows the relative error% in non-dimensional time calculations. By repeating the calculations with smaller integration step sizes, it is shown that the relative error depends on the integration step, such that smaller errors are achieved by using smaller integration step.

Figure 2. *Projectile motion in a vacuum: the relative error % in calculating time analytically using equation (28) and numerically using equation (19). The calculations are based on 0.001 radian step size*



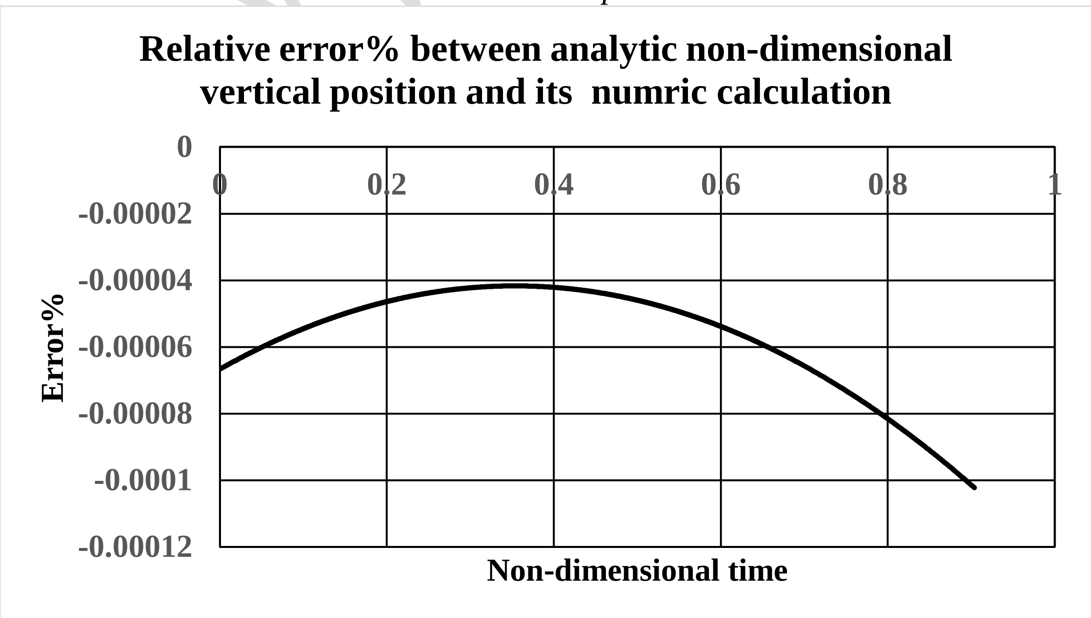
It is important to note that for the motion in a vacuum, the horizontal speed is a constant, thus the error in finding the x coordinate is the same as the error in calculating the time. This is depicted in figure 3.

Figure 3. *Projectile motion in vacuum: the relative error % in calculating horizontal position analytically using equation (23) and numerically using equation (20). The calculations are based on 0.001 radian step size*



Finally, the relative error % in calculating the non-dimensional vertical position is shown in figure 4.

Figure 4. *Projectile motion in vacuum: the relative error % in calculating vertical analytically using equation (25) and numerically using equation (21). The calculations are based on 0.001 radian step size*

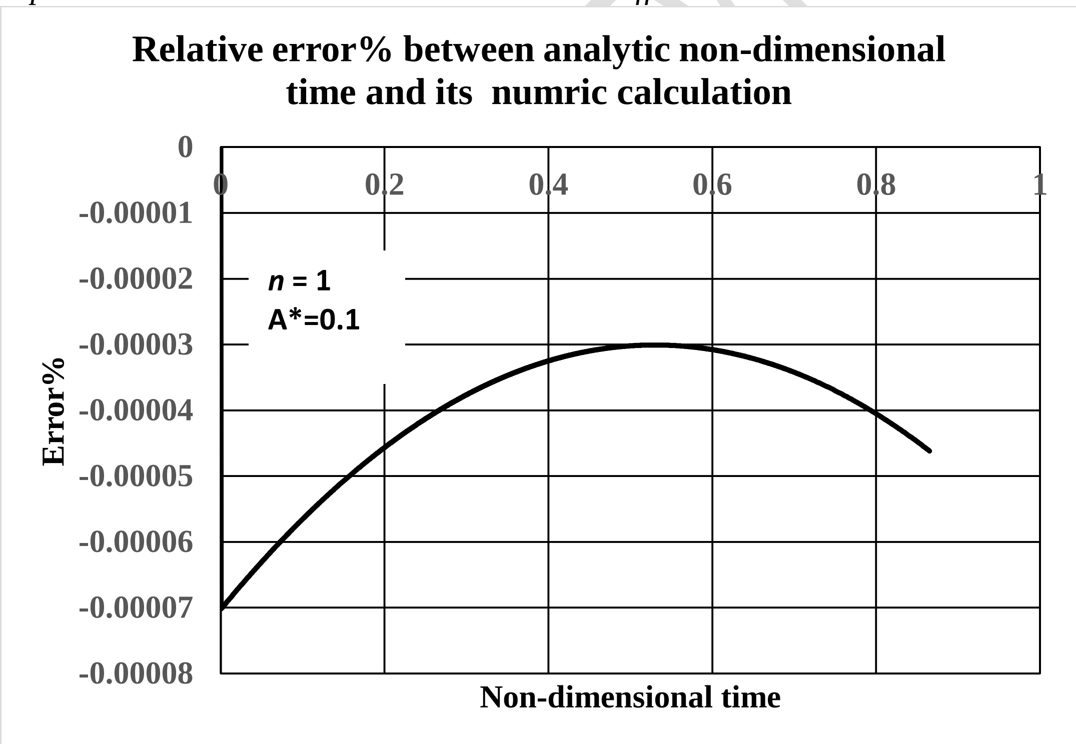


It is important to note that calculating the speed components were based on analytic formulas equations (22) and (27).

Linear air resistance model, n = 1

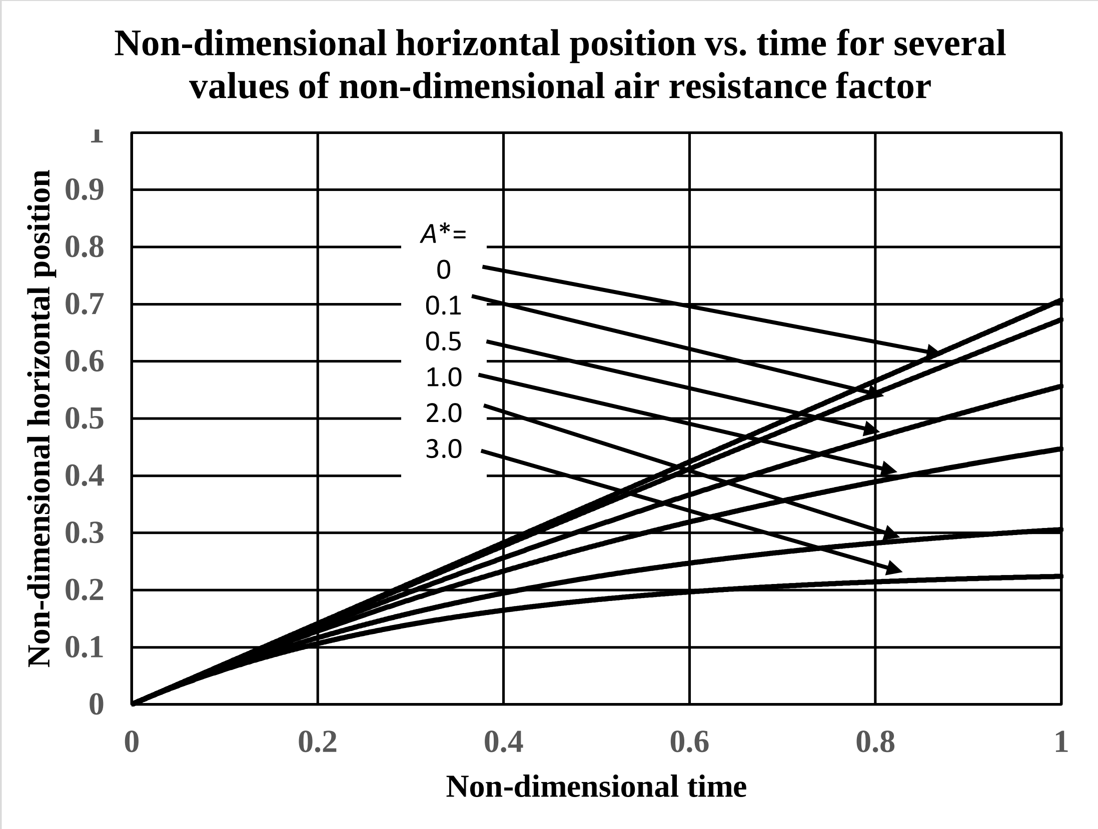
For the linear air resistance model with power $n = 1$ the time is calculated analytically and numerically. The relative error% is calculated and presented in figure 5. As was stated in the previous section, the accuracy depends on the step size of the numeric calculations. In fact, step size of 0.001 radian was used for the angle. In addition, the values of A^* affects the accuracy of the calculations; a fact that was verified numerically. Figure 5addresses the relative error% in calculating the non-dimensional time for the linear air resistance model with $n = 1$.

Figure 5. *Linear air resistance model with n = 1. The relative error % in calculating time analytically using equation (38) and numerically using equation (19). The calculations are based on 0.001 radian step size. The error depends on the no-dimensional air resistance coefficient A**



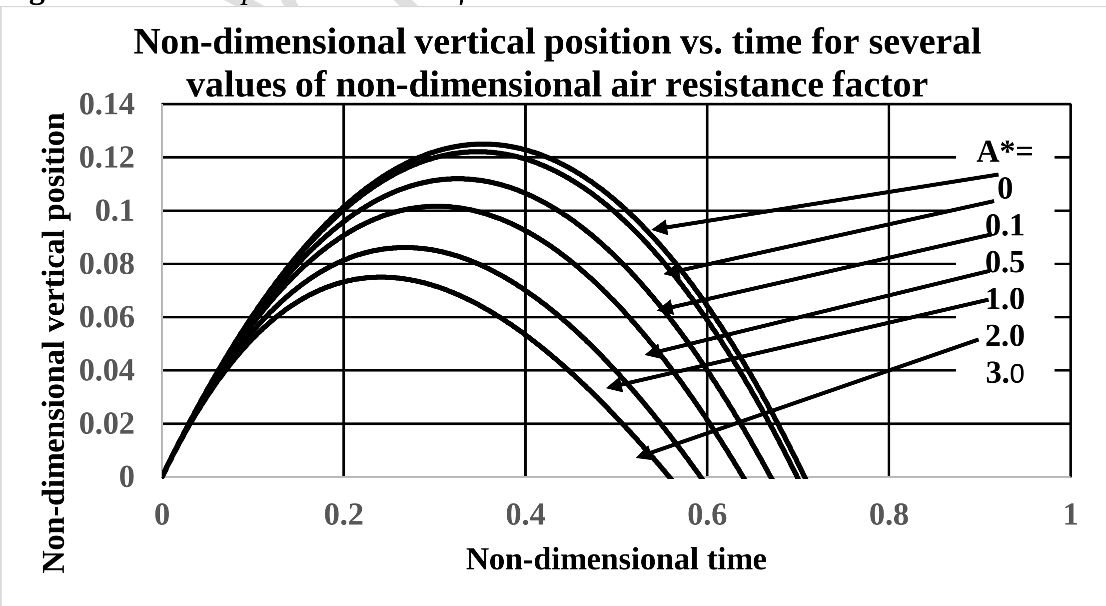
The projectile's position depends on the non-dimensional air resistance coefficient. Figure 6 shows the non-dimensional horizontal position as a function non-dimensional time.

Figure 6. *Horizontal position vs. time for the linear air resistance model*



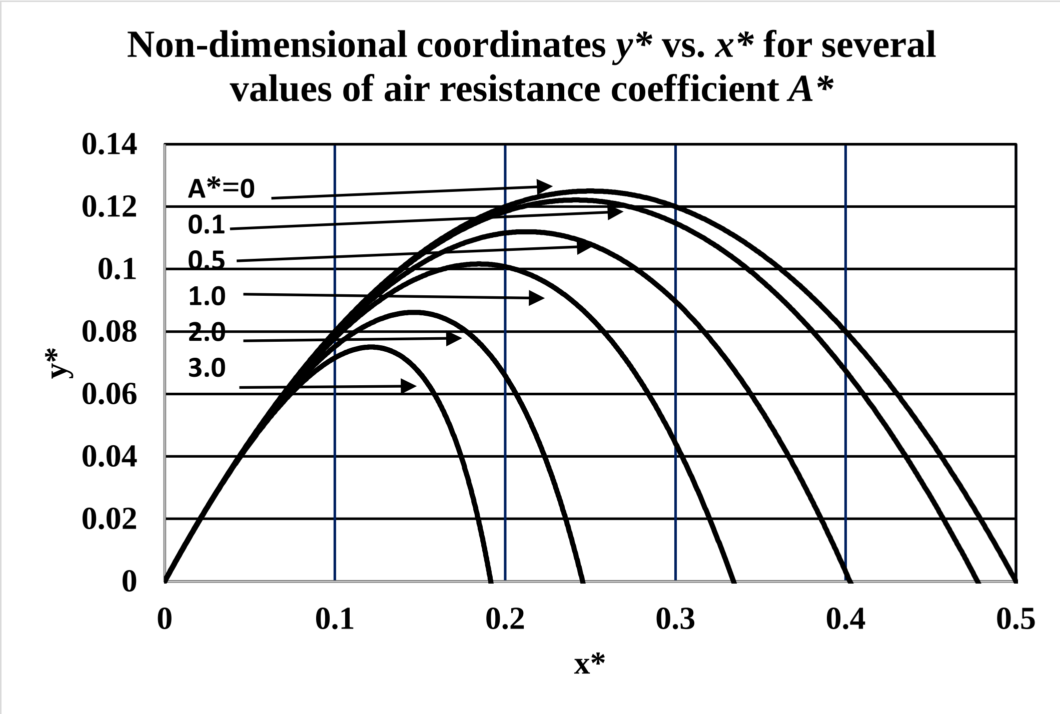
Similarly, Figure 7 shows the non-dimensional vertical position as a function non-dimensional time.

Figure 7. *Vertical position vs. time for the linear air resistance model*



Finally, the vertical position is plotted vs. horizontal position. It is important to note that this plot has a parabolic shape for a projectile's motion in a vacuum only, as was discovered by Galileo.

Figure 8. *Vertical position vs. horizontal position for the linear air resistance model*



As was pointed out early, the accuracy depends on the step size. In addition, the accumulated error depends on the length of flight time passed and the air resistance parameter. The following table shows relative error values calculated at angle -1 radian (see table 2). Maximum values are reached at the end of flight time interval.

Table 2. *Maximum relative error values (ME), T - time, X - horizontal position and Y - vertical position, for the linear model with n = 1*

A*	MET%	MEX%	MEY%
0.1	-4.61716E-05	-4.57806E-05	-9.36418E-05
10	-0.000240069	-0.000551681	-0.000956021
50	-0.002876372	-0.010973745	-0.013122853
100	-0.009489196	-0.04270971	-0.04747993
150	-0.019408972	-0.095145954	-0.102789086

It is clear from the table that the relative error increases with higher values of *A**, and increases with longer flight time.

Quadratic air resistance model, n = 2

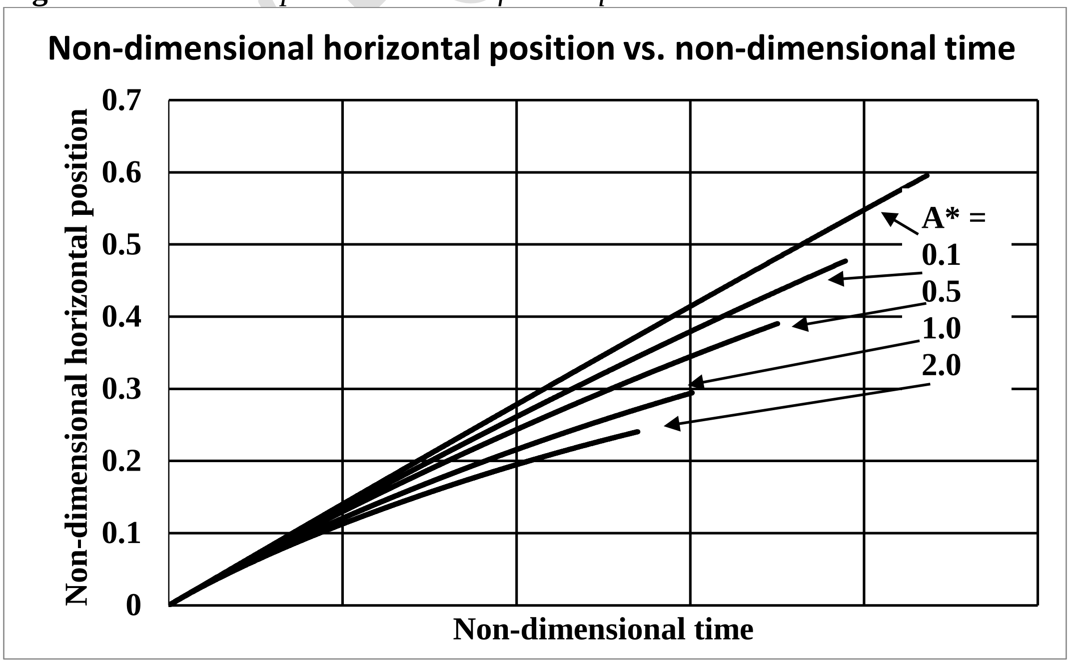
In the previous subsections, it was possible to compare analytic with numeric calculations. It is shown that good accuracy could be achieved by choosing proper integration step size. Although explicit formulas are not available, it might control the error by performing sensitivity analysis. For the general case, it is possible to solve the motion equations by performing different estimates. As was stated previously, error estimation could be based on comparison between semi-analytic calculations and numeric calculations. Based on this estimation method, the relative error in calculating the position coordinates is given in table 3.

Table 3. *Relative errors MEX% and MEY for quadratic air resistance model with n = 2*

*A**	MEX%	MEY%
0.1	2.26187E-06	6.2023E-06
10	2.65921E-05	1.82751E-05
50	3.49184E-05	3.69187E-05
100	3.77288E-05	4.13289E-05
150	3.91624E-05	4.33517E-05

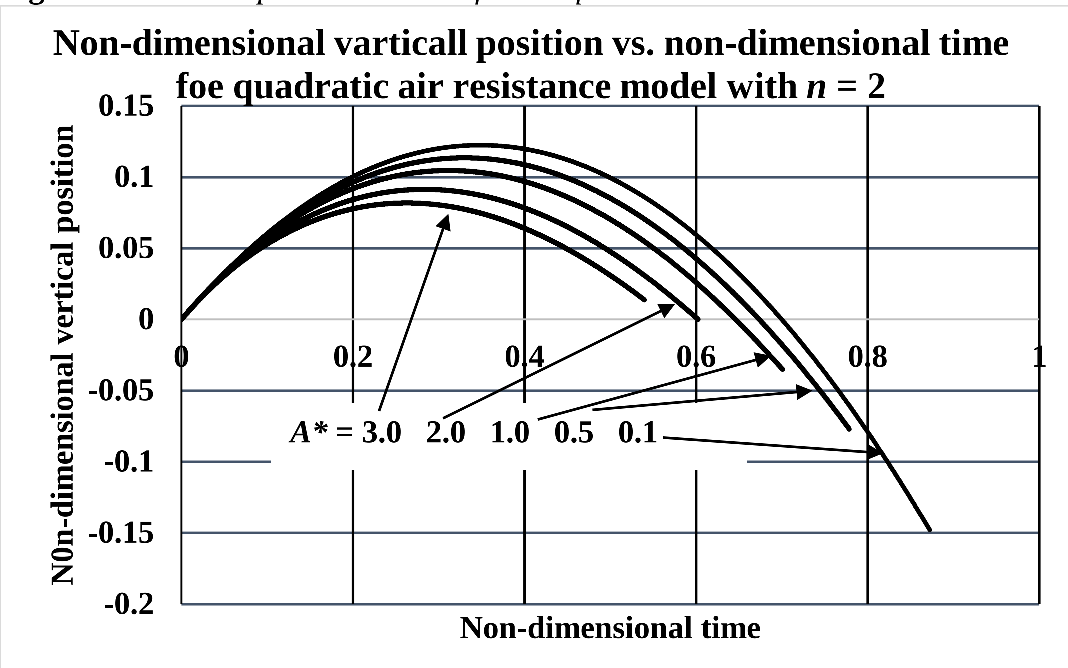
For illustrating the use of the general procedure, the non-dimensional horizontal coordinate is plotted vs. non-dimensional time for several values of the non-dimensional air resistance coefficient. It is observed that the horizontal coordinate is inversely related to friction with air (see figure 9).

Figure 9. *Horizontal position vs. time for the quadratic air resistance model*



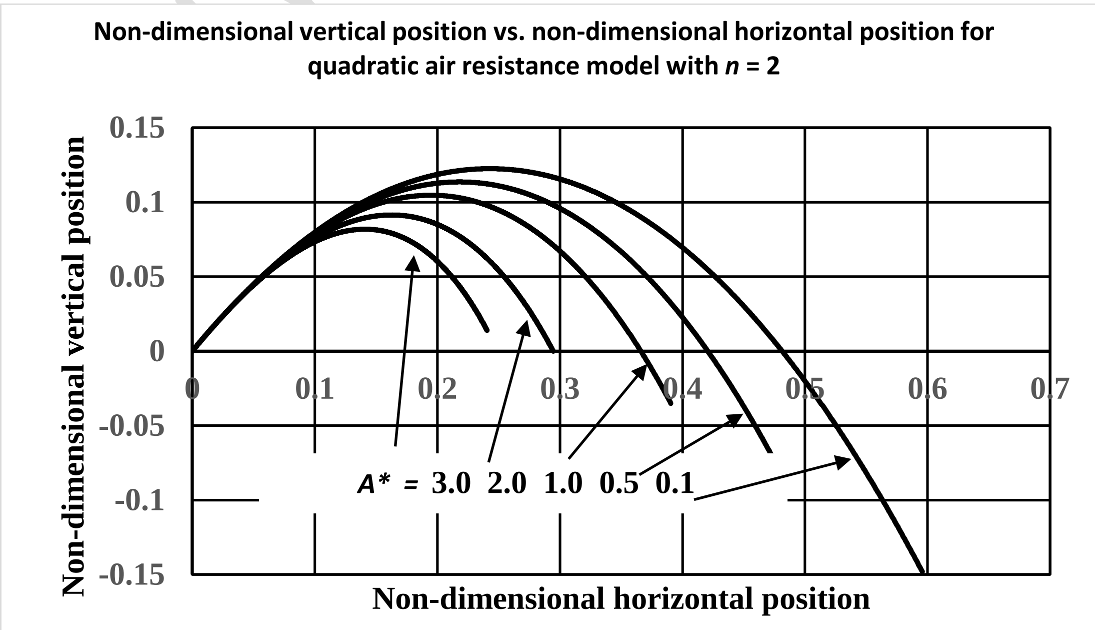
Similarly, the non-dimensional vertical coordinate is plotted vs. non-dimensional time for several values of the non-dimensional air resistance coefficient. Again, it is observed that the horizontal coordinate is inversely related to friction with air (see figure 10).

Figure 10. *Vertical position vs. time for the quadratic air resistance model*



Finally, the vertical coordinate is plotted vs. the horizontal coordinate to *n* = 2 (see figure 11).

Figure 11. *Vertical position vs. horizontal position for the quadratic air resistance model*



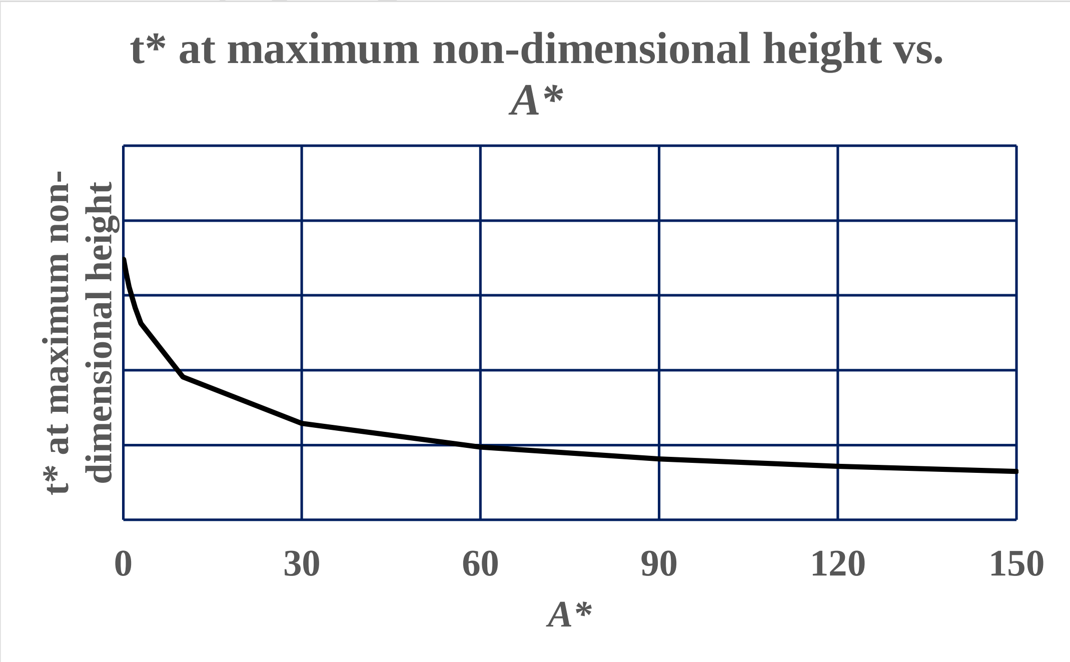
The general procedure could be used as a tool for education. For example, one might ask: "What is the time elapsed for reaching maximum height? "
By consulting equations (19) and (40), two limits are recognized: at zero *A**, the value of the ideal case is retrieved; and at large values of *A**, the time at maximum height is inversely related to the square root of *A**. Based on the spread sheet calculations the following results are achieved (see table 4):

Table 4. *Non-dimensional time as a function of non-dimensional air resistance coefficient for the case n = 2*

*A**	t*(max y*)
0.1	0.348183
0.5	0.329863
1	0.311316
2	0.283314
3	0.262672
10	0.191088
30	0.128932
60	0.097103
90	0.081528
120	0.071788
150	0.064941

For the convenience of the reader, the values given in table 4 are plotted in figure 12 (see figure 12).

Figure 12. *t* vs. A* for the case on n = 2*



Summary and conclusions

The projectile's motion with a general power law model of air resistance was studied using Cartesian coordinates. The equations of motion were derived from Newton's second law. The acceleration equations in the horizontal and vertical directions are non-linear and coupled ordinary differential equations.

The solution of these equations is simplified by a proper choice of the independent variables. In fact, decoupling the equations of motion was achieved by choosing the horizontal velocity and the angle of motion as independent variables. It is shown that the velocity components were derived analytically (see step 1 in the general procedure). The time (see step 2), the horizontal coordinate (see step 3) and the vertical coordinate (see step 4) were obtained by solving exact integrals. Trapezoidal rule quadrature was used to estimate the integrals.

Three cases of motion were considered: motion in a vacuum; motion under air resistance with $n = 1$; and motion under air resistance with $n = 2$.

The first two cases of motion were used to estimate the accuracy of the calculations. Furthermore, semi-analytic calculations and RK4 method were used to increase confidence in the accuracy estimates of the numerical calculations.

The time at maximum projectile's height was calculated for several values of the air resistance model for the case $n = 2$. It is shown that at large values of A^*, the time at maximum height is proportional to the inverse of the square root of A^*.

Finally, the general procedure of the solution of the projectile's motion could be used as a convenient educational tool for spreadsheet calculations.

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Projectiles with air resistance. https://dynref.engr.illinois.edu/afp.html, last visited on June 25th 2022.

Appendix 1

RK4 - model

```
Dim alfa As Double
Sub rk4()

Dim h, hf, t As Double
Dim x1, x2, x3, x4 As Double
Dim k1, k2, k3, k4 As Double
Dim l1, l2, l3, l4 As Double
Dim m1, m2, m3, m4 As Double
Dim n1, n2, n3, n4 As Double
x1 = 0
a = Atn(1)
MsgBox a
t = 0
x2 = Cos(a)
x3 = 0
x4 = Sin(a)
h = 0.001
hf = h / 2#
alfa = 3
For i = 0 To 2000
Cells(2 + i, 1) = t
Cells(2 + i, 2) = x1
Cells(2 + i, 3) = x2
Cells(2 + i, 4) = x3
Cells(2 + i, 5) = x4
k1 = h * x1dot(t, x1, x2, x3, x4)
l1 = h * x2dot(t, x1, x2, x3, x4)
m1 = h * x3dot(t, x1, x2, x3, x4)
```

```
n1 = h * x4dot(t, x1, x2, x3, x4)

k2 = h * x1dot(t + hf, x1 + k1 / 2, x2 + l1 / 2, x3 + m1 / 2, x4 + n1 / 2)
l2 = h * x2dot(t + hf, x1 + k1 / 2, x2 + l1 / 2, x3 + m1 / 2, x4 + n1 / 2)
m2 = h * x3dot(t + hf, x1 + k1 / 2, x2 + l1 / 2, x3 + m1 / 2, x4 + n1 / 2)
n2 = h * x4dot(t + hf, x1 + k1 / 2, x2 + l1 / 2, x3 + m1 / 2, x4 + n1 / 2)

k3 = h * x1dot(t + hf, x1 + k2 / 2, x2 + l2 / 2, x3 + m2 / 2, x4 + n2 / 2)
l3 = h * x2dot(t + hf, x1 + k2 / 2, x2 + l2 / 2, x3 + m2 / 2, x4 + n2 / 2)
m3 = h * x3dot(t + hf, x1 + k2 / 2, x2 + l2 / 2, x3 + m2 / 2, x4 + n2 / 2)
n3 = h * x4dot(t + hf, x1 + k2 / 2, x2 + l2 / 2, x3 + m2 / 2, x4 + n2 / 2)

k4 = h * x1dot(t + h, x1 + k3, x2 + l3, x3 + m3, x4 + n3)
l4 = h * x2dot(t + h, x1 + k3, x2 + l3, x3 + m3, x4 + n3)
m4 = h * x3dot(t + h, x1 + k3, x2 + l3, x3 + m3, x4 + n3)
n4 = h * x4dot(t + h, x1 + k3, x2 + l3, x3 + m3, x4 + n3)
t = t + h

x1 = x1 + (k1 + 2 * k2 + 2 * k3 + k4) / 6#
x2 = x2 + (l1 + 2 * l2 + 2 * l3 + l4) / 6#
x3 = x3 + (m1 + 2 * m2 + 2 * m3 + m4) / 6#
x4 = x4 + (n1 + 2 * n2 + 2 * n3 + n4) / 6#


Next i
End Sub
Function x1dot(t, z1, z2, z3, z4) As Double
x1dot = z2
End Function
Function x2dot(t, z1, z2, z3, z4) As Double
x2dot = -alfa * z2 * Sqr(z2 * z2 + z4 * z4)

End Function
Function x3dot(t, z1, z2, z3, z4) As Double
x3dot = z4
End Function
Function x4dot(t, z1, z2, z3, z4) As Double
x4dot = -2 - alfa * z4 * Sqr(z2 * z2 + z4 * z4)
End Function
```


Appendix 2

Based on [14] the following data are used:

Air density ρ =1.225 kg/m3.

Baseball mass *m* =0.145 kg.

Baseball diameter *D* = 0.075 m.

Air resistance coefficient: $c = \frac{1}{2} c_D \square \frac{D^2}{4}$

And the specific air resistance coefficient is calculated as $A = \frac{c}{m}$.

Reynolds number $Re = \frac{\rho v D}{\mu}$

(Density of air * speed * baseball diameter / dynamic viscosity of air).

For a speed of 40 m/s, the Reynolds number approximately 200000.

A^* = 3.0 (entry 7 in table 1).

In the numerical calculations we use A* values in the range 0. – 3.0.