

# Upgrading the Axiomatic System to 4-dimensional Space - axioms of Incidence

*There are three ways to approach the Euclidean geometry of four or more dimensions: axiomatic, algebraic (or analytical), and intuitive. Of course, only the first two can be formalized. The algebraic method is well developed and widely used, but this is not the case with the axiomatic. Although the idea of the higher-dimensional space was expressed by I. Kant and the properties have been explored by many mathematicians throughout history, there is no widely used axiomatic system. Here, following Somerville's ideas and a convenient axiomatic system for 3-dimensional space, we shall upgrade the axioms of incidence to 4-dimensional space.*

**Keywords:** *axiom, incidence relationship, hyperplane, hypercoplanar points*

## Introduction

Mathematic, or more precisely geometry, was the first science in which people found that it was necessary to develop a formal approach. It is the well-known Euclid's axiomatic system from the fifth century B.C. Over the centuries there have been many discussions about the postulates, changing and adding new ones.

In 1899, Hilbert (Hilbert 1957) introduced the first complete and consistent system of axioms of Euclidean geometry. Besides systems that are variations of this one, Birkhoff's axiomatic system for plane geometry (Birkhoff 1932) is also known, which uses real numbers and their properties. Nowadays, there are discussions about separate axioms, e.g. (Donnelly 18, Donnelly 19). More details on the development of the axiomatic system are given in e.g. (Wikipedia 2023a, Wikipedia 2023e).

The idea of the higher-dimensional space was expressed by I. Kant (1746), while J. d'Alembert (1764) wrote about connecting time with space as the fourth coordinate (Hazewinkel 1989). The first paper that explicitly deals with the geometry of  $n$  dimensions was that of A. Cayley in 1843. The importance of this subject was recognized by a group of British mathematicians: W.K. Clifford, J.J. Sylvester and German mathematicians: H.Grassmann, V. Schlegel, B. Riemann. Around 1852, a Swiss mathematician L. Schläfli completed his work on the "Theorie der vielfachen Kontinuität" (Schläfli 1901), but the first publication of the entire manuscript was in 1901, after Schläfli's death (Wikipedia 2022s). Some ideas about the axiomatic system in a higher dimension are given in (Gorn 1940) and (Wyler 1953).

In the beginning, the geometry of  $n$ -dimensional spaces was considered using a predominantly analytical (vector) method. In this paper, attention will be paid to the geometric point of view. Therefore Somerville's work (Somerville 1929) has great importance for us. As Coxeter (Coxeter 1973) wrote, Somerville provided

well founded geometrical approach and thanks to this it was later possible to develop geometry in  $n$ -dimensions (Banchoff 1990, Coxeter 1973).

In his book, Somerville (Somerville 1929) extended a series of terms for ‘linear spaces’: point, (straight) line, plane, with hyperplane or 3-flat, ...,  $m$ -flat. He expressed incidence structures using ‘linearly independent’ set of points. This means that he made a connection between vector spaces and geometric spaces, because each set of  $m+1$  points defines a set of  $m$  vectors (after choosing one of those points as the common beginning of all vectors, the other points would be the ends of these vectors), and vice versa.

In this paper, we shall mainly discuss the set of axioms of incidence in four dimensional space, starting from the system of axioms of incidence given in (Lučić 1994), as a variation of Hilbert’s system. Here we consider a possible upgrading of the set of axioms of incidence for 4-dimensional space. Finally, we give a brief discussion of other sets of axioms. Hints for further work and possible applications are also given.

### Used axioms of incidence in 3-dimensional space

Following Hilbert’s axiomatic system, several similar variations were made, e.g. (Lučić 1994, Paranjape 2001). Here we shall use and discuss the system of axioms given in (Lučić 1994) because in this book lines and planes are introduced by points, in an analogous way. Therefore, it would be easier to follow this method when introducing hyperplanes. This system also follows the Somerville’s method, so it will be simpler to check which statements hold for the incidence properties in 4-dimensional space.

The following is the system of incidence axioms in 3-dimensional space used here. For groups of the postulates, their purpose is given.

Incidence between points and lines:

**I(3)1.** Each line contains at least two distinct points.

**I(3)2.** There is at least one line that contains two points.

**I(3)3.** There is at most one line that contains two distinct points.

Incidence between points and planes:

**I(3)4.** Each plane contains at least three non-collinear points.

**I(3)5.** There is at least one plane that contains three points.

**I(3)6.** There is at most one plane that contains three non-collinear points.

Incidence between lines and planes:

**I(3)7.** If a line contains two distinct points that lie on a plane, then every point on that line lies on the plane.

Dimensionality of space:

**I(3)8.** If two distinct planes contain a common point, then they contain at least one other common point.

**I(3)9.** There are at least four non-coplanar points.

The terms used are defined below.

**Definition 1.** *Collinear points* are those that lie on the same line.

**Definition 2.** A line lies in a plane if all points lying in that line also lie in the plane.

**Definition 3.** Coplanar points are those that lie on the same plane.

In the same book (Lučić 1994) the following corollaries of the used axioms of incidence are listed. Since the proofs are simple, they are omitted, both in the mentioned book and here.

**Corollary 1.** If three points are non-collinear, then every two of them are mutually different.

**Corollary 2.** If four points are non-coplanar, then every two of them are mutually different and every three of them are non-collinear.

**Corollary 3.** There are three non-collinear points.

**Corollary 4.** There is a unique line that contains two different points.

**Corollary 5.** There is a unique plane that contains three non-collinear points.

**Corollary 6.** There is a unique plane that contains a line and a point that does not belong to it.

**Corollary 7.** There is a unique plane that contains two different intersecting lines. We need the following definition.

**Definition 4.** Two lines lying in different planes are said to be *skew lines*.

Then we conclude that the next corollary holds.

**Corollary 8.** There are two skew lines.

Other listed corollaries we mentioned are:

**Corollary 9.** The intersection of two different lines is at most one point.

**Corollary 10.** The intersection of a line and a plane that does not contain it is at most one point.

**Corollary 11.** If two different planes have a common point, their intersection is a line.

**Corollary 12.** There are four different points, six different lines, four different planes.

## Introducing the axioms of incidence for 4-dimensional space

There are several reasons to introduce a system of axioms for 4-dimensional space. Besides historical reasons and the possibility of easier research of Euclidean and hyperbolic geometry, an important reason is that vectors allow introducing only direct isometries.

Here, a system of axioms of incidence for 4-dimensional space will be introduced, through consideration of the purpose of individual postulates.

### *Incidence between Points and Hyperplanes*

First, we need to introduce a 3-dimensional linear subspace of the entire 4-dimensional space. Following (Somerville 1929), let us introduce the term *hyperplane* for a 3-dimensional linear subspace. It was mentioned earlier that the hyperplane can be introduced similarly as it was done with the line by I(3)1 - I(3)3 and with the plane by I(3)4 - I(3)6. We shall keep the postulates I(3)1 - I(3)6 with the new notations **I(4)1 - I(4)6** as part of the system of incidence axioms of 4-

dimensional space. Note that Corollaries 1-10 are still valid as consequences of I(4)1 - I(4)6.

Then the incidence between points and hyperplanes would be introduced with following postulates:

**I(4)7.** Each hyperplane contains at least four non-coplanar points.

**I(4)8.** There is at least one hyperplane that contains four points.

**I(4)9.** There is at most one hyperplane that contains four non-coplanar points.

Besides the previous Definitions, we also need the following.

**Definition 5.** A line lies in a hyperplane if all points lying in that line also lie in the hyperplane.

**Definition 6.** A plane lies in a hyperplane if all points lying in that plane also lie in the hyperplane.

The direct Corollary that follows from postulates I(4)7 - I(4)9 and which we need in further consideration is:

**Corollary 13.** There is a unique hyperplane that contains four non-coplanar points.

As before, we shall omit proofs of simple Corollaries here.

### *Incidence between Lines, Planes and Hyperplanes*

In describing the relationship between linear spaces of different dimension, we have to be careful. Besides the postulate I(3)7 concerning the relationship between a line and a plane, we have to express the relations between a line and a hyperplane, and between a plane and a hyperplane either within the axiomatic system or as a corollary. We can conclude that it would be enough to keep the postulate I(3)7 with the new notation **I(4)10**, and to add one of the statements either S(4)1 or S(4)2. These statements are as follows:

**S(4)1.** If a line contains two distinct points which lie on a hyperplane, then every point on that line lies on the hyperplane.

**S(4)2.** If a plane contains three non-collinear points which lie on a hyperplane, then every point on that plane lies on the hyperplane.

We shall consider how one statement leads to another. As S(4)1 looks simpler, we shall prove Theorem 1 first.

**Theorem 1.** If postulates I(4)1 - I(4)10, axioms of order and S(4)1 hold, then S(4)2 holds.

In following proof, we fundamentally need axioms of order. They are devoted to the relationship 'between' (Hilbert 1957, Lučić 1994). In this sense, we say that point  $C$  is between points  $A$  and  $B$  and write  $\beta(A, C, B)$ . From this system of axioms we need Pasch's postulate and a corollary of axioms of order which is denoted as Theorem 2.4 in (Lučić 1994) and here with Claim 1.

**Pasch's postulate.** If  $A, B, C$  are three non-collinear points and if  $p$  is a line in the plane  $ABC$  which does not meet point  $C$  and intersects line  $AB$  at point  $P$  such that  $\beta(A, P, B)$ , then  $p$  intersects either line  $AC$  in point  $Q$  such that  $\beta(A, Q, C)$  or line  $BC$  in point  $R$  such that  $\beta(B, R, C)$ .

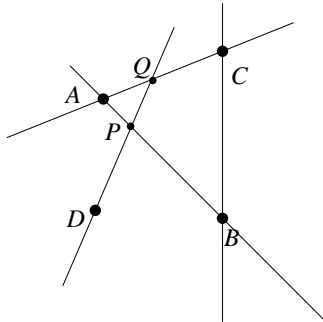
**Claim 1.** If  $A, B$  are two distinct points, then there exists a point  $C$  such that  $\beta(A, C, B)$ .

The proof of Theorem 1 follows.

**Proof.** Let  $A, B, C$  be non-collinear points lying on the plane  $\alpha$  and on the hyperplane  $\Pi$ . According to I(4)10 and S(4)1, every point  $E$  that lies on some of the lines  $AB, AC, BC$  also lies on  $\alpha$  and on  $\Pi$ . Let  $D$  be any point in  $\alpha$  that does not lie on the lines  $AB, AC, BC$  and let  $P$  be any point lying on the line  $AB$  such that  $\beta(A, P, B)$ . According to Claim 1, such a point  $P$  exists. We have two possible cases.

1. If  $C \in DP$  then there are two distinct points  $P$  and  $C$  of  $DP$  lying on  $\Pi$ . According to S(4)1, all points from  $DP$  lie on  $\Pi$ . So,  $D$  lies on  $\Pi$ .

**Figure 1.** Proof of Theorem 1, case 2



2. When  $C \notin DP$ , by Pasch's postulate,  $DP$  either intersects line  $AC$  in  $Q$  such that  $\beta(A, Q, C)$  or  $BC$  in  $R$  such that  $\beta(B, R, C)$  (see Figure 1). Then the points  $P, Q$  or  $P, R$  are two different points from  $DP$  lying on  $\Pi$ . Again by S(4)1,  $D$  lies on  $\Pi$ .

□

In the opposite direction we have Theorem 2.

**Theorem 2.** If postulates I(4)1 - I(4)10 and S(4)2 hold, then S(4)1 holds.

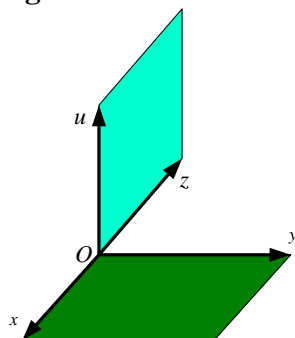
**Proof.** Let  $A$  and  $B$  be two distinct points lying on the line  $p$  and on the hyperplane  $\Pi$ . Let  $C$  be a point that lies on  $\Pi$  and is non-collinear with points  $A, B$ . Then  $A, B, C$  according to Corollary 6, defines the plane  $\alpha$ . Therefore, plane  $\alpha$  and hyperplane  $\Pi$  have three common non-collinear points, so according to S(4)2 every point of plane  $\alpha$  lies on hyperplane  $\Pi$ . Also, since the line  $p$  and the plane  $\alpha$  have common points  $A$  and  $B$ , according to I(4)10 every point of the line  $p$  lies on the plane  $\alpha$ . Consequently, every point of the line  $p$  lies on the hyperplane  $\Pi$ . □

As for the proof of Theorem 2, unlike the proof of Theorem 1, no axioms of order are required, but only axioms of incidences, we shall use S(4)2 as postulate I(4)11. S(4)1 will be our **Corollary 14**.

### Dimensionality of Space

Postulates I(3)8 and I(3)9 remain to be discussed. First, let's note that in 4-dimensional space I(3)8 does not hold! We can easily see this fact if we consider the intersection of the  $Oxy$  and  $Ozu$  coordinate planes in the 4-dimensional Cartesian coordinate system. Their only common point is  $O$ .

**Figure 2.** Two Planes in 4-dimensional Space with only one Common Point



We can conclude that the purpose of postulate I(3)8 in 3-dimensional space is to give, in Somerville terms (Somerville 1929), an upper bound on the number of linearly independent points for the entire (3-dimensional) space. It is not so difficult to observe that in 4-dimensional space we also need an upper bound on the number of linearly independent points for the entire space, but also for linear subspaces. The first thing we need in 4-dimensional space is to modify the postulate I(3)8 into I(4)12.

**I(4)12.** If two distinct planes *lying in the same hyperplane*, contain a common point, then they contain at least one other common point.

Besides this postulate the postulate I(4)13 is also required.

**I(4)13.** If a hyperplane and a plane contain a common point, then they contain at least one other common point.

It is obvious that in 4-dimensional space we need more linearly independent points than in 3-dimensional space. Since the hyperplane is determined by four non-coplanar points, we need at least one more point for entire 4-dimensional space, which is linearly independent of the four non-coplanar ones used previously. This is also consistent with (Somerville 1929). More precisely, we can define

**Definition 6.** *Hypercoplanar points* are those that lie on the same hyperplane.

We also observe that the postulate I(3)9 has to be replaced by a new one.

**I(4)14.** There are at least five non-hypercoplanar points.

Based on everything said so far, an upgraded system of axioms of incidence for 4-dimensional space consists of

**I(4)1.** Each line contains at least two distinct points.

**I(4)2.** There is at least one line that contains two points.

**I(4)3.** There is at most one line that contains two distinct points.

**I(4)4.** Each plane contains at least three non-collinear points.

**I(4)5.** There is at least one plane that contains three points.

**I(4)6.** There is at most one plane that contains three non-collinear points.

**I(4)7.** Each hyperplane contains at least four non-coplanar points.

**I(4)8.** There is at least one hyperplane that contains four points.

**I(4)9.** There is at most one hyperplane that contains four non-coplanar points.

**I(4)10.** If a line contains two distinct points that lie on a plane, then every point on that line lies on the plane.

**I(4)11.** If a plane contains three non-collinear points which lie on a hyperplane, then every point on that plane lies on the hyperplane.

**I(4)12.** If two distinct planes lying in the same hyperplane, contain a common point, then they contain at least one other common point.

**I(4)13.** If a hyperplane and a plane contain a common point, then they contain at least one other common point.

**I(4)14.** There are at least five non-hypercoplanar points.

Some of the new Corollaries that follow from postulates I(4)1 - I(4)14 are:

**Corollary 15.** If five points are non-hypercoplanar, then every two of them are mutually different, every three of them are non-collinear, and every four of them are non-coplanar.

**Corollary 16.** There are four non-coplanar points.

**Corollary 17.** The intersection of a line and a hyperplane that does not contain it is at most one point.

**Corollary 18.** There is a unique hyperplane that contains four non-coplanar points.

**Corollary 19.** There is a unique hyperplane that contains two skew lines.

**Corollary 20.** There is a unique hyperplane that contains a plane and a point that does not belong to it.

**Corollary 21.** There is a unique hyperplane that contains a plane and a line whose intersection is a point.

**Corollary 22.** There is a unique hyperplane that contains two different planes whose intersection is a line.

We need the next term.

**Definition 7.** A line and a plane which do not belong to the same hyperplane are said to be *skew*.

Then the following corollary is true.

**Corollary 23.** There are a line and a plane that are skew.

The Corollaries 11 and 12 are no longer correct and should be modified.

**Corollary 11\*.** If two different planes lying in the same hyperplane have a common point, their intersection is a line.

**Corollary 12\*.** There are five different points, ten different lines, ten different planes, five different hyperplanes.

## Discussion of other Groups of Axioms and Hints for the Further Work

Axiom systems can also be considered in dimensions greater than four. In addition to considering the axiom of incidence, this includes other groups of axioms, as well as different approaches to the axiomatic system. Therefore, we shall mention here the following research possibilities.

### *Axioms of Order*

In this group of axioms there are six postulates, five of which refer to the order of points in a straight line, while the sixth - Pasch's postulate refers to points in a plane (Lučić 1994, Hilbert 1957). Even in axiomatic system of 3-dimensional space there are no postulates about points in dimension three. Thus, it is expected that it

would be sufficient to work with existing set of axioms of order in higher dimensions.

Indeed, the terms half-line, half-plane, half-space are introduced similarly, taking into account whether the points are ‘on the same side’ or ‘on different sides’ of a single point, line, plane. The similarity in approach is also evident in the following discussion of different dimensions (up to three) in other consequences of the axioms of order (Lučić 1994). So, we expect that the similarity in introducing terms and proving theorems can be followed in higher dimensions without adding new postulates.

That may be the subject of some future paper.

### *Axioms of Congruence*

In this set of axioms, in each of the individual postulates, the set of points at the origin is either on a line or in a plane. This is also true for the set of compared points, but the observed dimension is not precisely specified anywhere. Again, it is to be expected that there is no need for new postulates in higher dimensions.

These first three systems of axioms together enable the introduction of geometric objects such as polyhedra, polytopes (Coxeter 1973, Banchoff 1990) and their projection into 2-plane (Katona et al. 2011), as well as isometries and isometry groups, as an important part of geometric research (Stojanović 2017, Szirmai 2018, Molnár et al. 2023). Therefore, their more detailed consideration would be of great importance.

### *Continuity and Parallelism*

The axioms of continuity and parallelism can be considered independently of other axiom systems and based on that fact, special types of geometries such as finite geometries (Jinan 2022, Knill 2023) and hyperbolic (Lobachevsky - Bolyai) geometry were developed (Stojanović 2017, Szirmai 2018, Molnár et al. 2023). It is therefore obvious that the same axioms could be used in higher dimensions, without modification.

Moreover, the axioms of continuity talk only about points on a line. Similarly, the axiom of parallelism, of both Euclidean and hyperbolic space, is about ‘parallel lines’ in some plane.

Considering all the axioms in higher dimensions together (except perhaps the axiom of parallelism) would allow us to introduce  $n$ -dimensional vectors and prove the completeness and non-contradiction of the introduced system of axioms.

### *Using Computer Programs in testing Sets of Axiom*

It is interesting to mention at the end of this paper that the development of computer technology and programs enables us to generate and verify theorem proofs. This was done for some geometric theorems in (Stojanović-Đurđević 2016,

Stojanović-Đurđević 2019) with the usual axiomatic system of 3-dimensional space as input data to the program.

Similar programs can be used in the future to validate sets of axioms in dimension four and higher, as well as to prove their consequent theorems.

## References

- Banchoff T F (1990) *Beyond the third dimension* The Scientific American Library, New York.
- Birkhoff G D (1932) A set of postulates for plane geometry based on scale and protractor *Ann. Math.* 33(2): 329–345.
- Coxeter H S M (1973) *Regular Polytopes* 3rd. ed., Dover, New York.
- Donnelly J (2018) A Model of Nowhere Geodesic Plane Geometry in which the Triangle Inequality Fails Everywhere *Forum Geometricorum* 18: 275–296.
- Donnelly J (2019) The non-redundancy of side-angle-side as an axiom of the non-continuous non-euclidean plane *International journal of geometry* 8(2):70–83
- Gorn S (1940) On incidence geometry *Bull. Amer. Math. Soc.* 46: 158–167.
- Hazewinkel M (1989) *Encyclopedia of Mathematics* Vol.4, Kluwer Academic Publishers.
- Hilbert D (1957) *Osnove geometrije* Matematički Institut, Beograd (origin: *Grundlagen der Geometrie*, 1899.)
- Jinan J (2022) Some models of the finite hyperbolic geometry and the finite hyperbolic plane *ANJS* 25(4): 59–62  
DOI: 10.22401/ANJS.25.4.10
- Katona J, Molnár E, Prok I, Szirmai J (2011) Higher-dimensional central projection into 2-plane with visibility and applications *Kragujevac Journal of Mathematics* 35 (2): 249–263.
- Knill O (2023) Finite topologies for finite geometries *Preprint*  
DOI: 10.48550/arXiv.2301.03156
- Lučić Z (1994) *Euklidska i hiperbolicka geometrija* Grafiti i Matematički fakultet.
- Molnár E, Stojanović M, Szirmai J (2023) Non-fundamental trunc-simplex tilings and their optimal hyperball packings and coverings in hyperbolic space, I. For families F1-F4 *Filomat* 37(5): 1409–1448  
DOI: 10.2298/FIL2305409M
- Paranjape K (2001) The Axioms of Incidence *The Institute of Mathematical Sciences (IMSc)* <https://www.imsc.res.in/~kapil/geometry/euclid/node2.html>
- Schläfli L (1901) *Theorie der vielfachen Kontinuität* Springer Basel AG
- Somerville D M Y (1929) *An introduction to the geometry of n dimensions* London.
- Stojanović-Đurđević S (2016) Automated verification of informal proofs from high school geometry (in Serbian) *InfoM* 58: 11–19.
- Stojanović-Đurđević S (2019) From informal to formal proofs in Euclidean geometry *Ann Math Artif Intell* 85: 89–117. <https://doi.org/10.1007/s10472-018-9597-7>
- Stojanović M (2017) Hyperbolic space groups and their supergroups for fundamental simplex tilings *Acta Mathematica Hungarica* 153(2): 276–288. DOI: 10.1007/s10474-017-0761-z
- Szirmai J (2018) Horoball packings related to the 4-dimensional hyperbolic 24 cell honeycomb  $\{3,4,3,4\}$  *Filomat* 32(1):87-100. DOI: 10.2298/FIL1801087S
- Wikipedia (2022s) *Ludwig Schläfli* available at [https://en.wikipedia.org/wiki/Ludwig\\_Schläfli](https://en.wikipedia.org/wiki/Ludwig_Schläfli)

- 1   Wikipedia (2023a) *Axiom* available at <https://en.wikipedia.org/wiki/Axiom>
- 2   Wikipedia (2023e) *Euclidean geometry* available at [https://en.wikipedia.org/wiki/Euclidean\\_geometry](https://en.wikipedia.org/wiki/Euclidean_geometry)
- 3       *dean\_geometry*
- 4   Wyller O (1953) Incidence geometry *Duke Math. J.* 20: 601–610.
- 5

ONLY FOR REVIEW