A Note on Optimal Time for Closing a Trading Position

In this paper, trading rules (strategies) on a specified financial asset at some future time are interpreted as contingent claims (financial derivatives). Therefore, their fair values are computable using the binomial tree technique. However, traders pay the price of financial asset at the current time to enter to trading. Clearly, it is a loss for traders. In this paper, first, hedging strategies are proposed. Then, using three procedures the optimal time for closing the trading position are derived. Mentioned procedures are based on optimal stopping time and stochastic dynamic programming, state space and a practical procedure which uses an adds-in of Excel software. Indeed, optimal closing time and related trading strategies are applied in discrete time price processes and in the binomial tree setting. Markov decision process (MDP) solution to the problem is proposed. Simulation results are studied and finally, a conclusion section is given.

Keywords: *Binomial tree; Fair value; Financial derivative; Excel; Hedging; MDP; Optimal stopping; State space model; Stochastic dynamic programming; Trading strategies*

Introduction

Trading is action of buying and selling financial assets in any financial markets to gain for himself or for any other person or firm. Some typical financial assets are stocks, equities, shares, exchange rates (forex), derivatives like options, futures, forward, swaps, and recently crypto-currencies such as bit-coin Traders can be considered as an investor which holds asset in a short duration. There are many technical concepts related to trading such as volume, standards, business, account of trading. Also, trading has many formats such as insider, day and intraday, fair, swing, Duluth, online, binary and momentum trading versions. There are many types of orders in trading such as market order, limit order, stop order, stop-limit, day, good-till-cancelled, immediate-or-cancelled, fill-or-kill and all-or-none orders.

Traders bet on future value of financial asset such as stock. Suppose that, in the current time $t = 0$, the price of financial asset is s_0. Traders forecast s_T, the price of financial asset at some future time $T > 0$, and based on their forecasts $\hat{s}_T$, they do their trades including buying or selling. They use trading rule $X = f_T(s_T)$. Indeed, the trading is a kind of betting in future prices of financial assets and therefore it is a kind of contingent claim (financial derivative). However, there is a contradiction, as follows.

The fair price of trading rule X at $t = 0$ is

$$f_0 = e^{-rT} E_Q(X|F_0),$$

where r and Q are the risk free rate and the risk neutral probability measure equivalent to physical measure P which governs on s. Here, the σ-field F_0

contains all information of trader at time zero. Trading strategies usually contain hedging strategies at maturity T, to avoid bad probable events. Thus, it is natural to assume that $f_T = f(s_T) \leq s_T$. Hence, using monotone property of expectation, it is seen that $f_0 \leq s_0$ at which this is the loss of trader. Indeed, the trader pays $s_0 \geq f_0$ to enter the trade but he/she gains$f_T \leq s_T$. Let $L_T = f_T - s_T$ denote the loss of trader at maturity. In this paper, it is interested to find the stopping time τ_*, the minimizer of L_{τ_*}. At τ_*, the trader closes the trading position.

As an example, consider the simple stop-loss strategy at which the trader orders to his/her broker to sell the financial asset at the price of s_T if the trace of price be increasing in time and s_T are bigger than the threshold m. Conversely, if the map of price is decreasing in time and it is expected that s_T will be less than m, then trader orders to his/her broker to sell financial asset at price m. Therefore,

$$f(s_T) = \begin{cases} s_T & s_T > m \\ m & s_T \leq m \end{cases}$$

Equivalently, $f(s_T) = \max(s_T, m) = \max(s_T - m, 0) + m$. The fair price of this trading strategy is the price of a call option with strike price m and a bond with face value m. Here, it is interested to find a stopping time τ_* to minimize

$$min_{0\leq\tau\leq T} E_Q(L_\tau|F_0) = -max_{0\leq\tau\leq T} E_Q(-L_\tau|F_0) = -max_{0\leq\tau\leq T} E_Q(s_\tau - f_\tau).$$

The s_0 is kept fixed as a non-random variable. This is a standard problem of optimal stopping techniques; see Shiryaev and Novikov (2008). In this paper, it is aimed to characterize the optimal closing time (stopping time) of a specified trading strategies to reduce the overall loss of trader. This kind of stopping time is derived for discreet time trading strategy. Then, hedging strategies are given. The state space formulation is proposed and the binomial tree version is studied. MDP solutions are given. Finally, a conclusion section is also given.

Optimal closing time

In this section, discrete time trading strategy based a binomial tree setting is studied and optimal closing time is obtained in constant and time varying volatilities cases. Assume that $s_k = s_{k-1}x_k$ where x_k's are independent and identically distributed and suppose that there are k days to maturity. Suppose that x_k is u with probability of p and d with probability of $1-p$. Here, to avoid arbitrage opportunities, it is assumed that $d < e^r < u$, at which r is a risk free rate. The risk neutral probability measure is given by $Q{:}\,(p_{rn}, 1 -$

$p_{rn}), p_{rn} = \frac{e^r - d}{u-d}$, see Bjork (2009). Here, rn stands for risk neutral probability measure. The main tool for solving optimal closing time (optimal stopping) is the dynamic programming, see Tijms (2012).

Constant Volatility

Here, assuming the constant volatility, the optimal closing time of trading for trader is derived. Let the current price of financial asset be s. Following Shiryaev and Zhitlukhin (2013), the dynamic programming based backward induction implies that

$$V_k(s) = \min\left(s - f, E_Q(V_{k-1}(sx_k)|F_k)\right), k = T, \dots, 1,$$

such that $V_0 = s - f$. The optimal time for closing the trading position is given as follows

$$\tau_* = \inf\{k, V_k(s) = s - f\}.$$

It is seen that τ_* is an early exercise time of an American type of financial derivative with pay-off function $f(s_T)$ at the maturity T which is an interesting result. Indeed, the stopping time τ_* can be determined in a binomial tree. As follows, a theoretical procedure based a stochastic dynamic programming is proposed.

<u>*Procedure 1*</u>. Here, using approach of Ross (1982), pages 4,5, a solution is proposed. First, notice that

$$V_k(s) = \min\left(s - f, p_{rn}V_{k-1}(su) + (1 - p_{rn})V_{k-1}(sd)\right), V_0(s) = s - f.$$

Let $U_k(s) = V_k(s) - s$. Then,

$$U_k(s) = \min(-f, p_{rn}U_{k-1}(su) + (1 - p_{rn})U_{k-1}(sd) + sA), U_0(s) = -f,$$

where $A = p_{rn}(u - 1) + (1 - p_{rn})(d - 1) = e^r - 1$. Following Ross (1982), $U_k(s)$ is decreasing in s. The proof is by induction on k. Thus, the optimal policy has the following form. To this end, suppose that the current price is s and there are k days to maturity.

Proposition 1 (Optimal policy). Suppose that there are increasing numbers $s_1 < \cdots < s_n < \cdots$, then one should close the trading if and only if $s_n \leq s$.

In the rest of this section, two other procedures are proposed. The procedure 2 contains a practical solution based on Excel software.

<u>*Procedure 2*</u>. Some adds-in of Excel software such as *DerivaGem*[1] derive the fair price and early exercise time of some specified American type financial derivatives such as call or put options. However, a difficulty of this approach is that sometimes the specific financial derivative (trading strategy) is combination of some financial derivatives. The *DerivaGem* software specifies the early exercise for each component separately but the early exercise of

[1]See http://www.prenhall.com/mischtm/support_fr.html.

combination of financial derivatives is unknown, yet. A natural question is how to modify the *DerivaGem* to value and show the early exercise for every arbitrary derivative? The answer is so simple. It is enough to find the early exercise of each component, separately, and then consider the common early exercises, as early exercise of financial derivative (trading strategy).

State Space Modeling

The discussion of part 2.1 relies on strong assumption of constant volatility, which is not correct in practice. To overcome this difficulty, Liao (2005) considered a GARCH(1,1) series for squared volatility $h_t = v_t^2$ of financial asset. This equation plays the role of state equation in a state space modeling which leads to a Bayes filtering approach. Here, following Liao (2005), assume that $f_t = f_t(s_t, v_t)$ represent the binomial tree (or Black-Scholes, hereafter BS) price of a financial derivative. However, because of wrong assumption of constant volatility, there is a deviation ε_t for price computed using standard BS formula. Thus,

$$f_t = BS(v_t) + \varepsilon_t.$$

Here, ε_t's are independent and identically distributed random variables with common distribution with zero mean and variance σ_ε^2. This equation plays the role of measurement equation of state space model. Let h_t be an GARCH(1,1) series given by

$$h_\mathrm{t} = \omega + \alpha \mathrm{r}_\mathrm{t}^2 + \beta h_{\mathrm{t}-1} + \zeta_t.$$

This equation plays the role of state equation. Here, it is assumed that ζ_t's are independent and normally distributed random variables with zero mean and variance σ_ζ^2. It is assumed that ε_t's and ζ_t's are statistically independent. Also, assumed that $\omega, \alpha, \beta > 0$ and $\alpha + \beta < 1$. This section can be considered as the Liao (2005) work in European derivatives to American format. Here, r_t's are returns of financial asset which is underlying asset and derivative is defined basis on it.

As follows, based on Bayes rule, updating procedures are derived. Notice that $f_t = BS(v_t) + \varepsilon_t = BS^*(h_t) + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, $BS^*(x) = BS(\sqrt{x}\ )$, $h_t = v_t^2, \mu_t = BS^*(h_t)$. Also, it is known that $h_\mathrm{t} = \omega + \alpha \mathrm{r}_\mathrm{t}^2 + \beta h_{\mathrm{t}-1} + \zeta_t$ and $\zeta_t \sim N(0, \sigma_\zeta^2)$. Thus, $f_t | h_t \sim N(\mu_t, \sigma_\varepsilon^2)$ and given $h_{\mathrm{t}-1}$ and r_t, then $h_t \sim N(\theta_t, \sigma_\zeta^2)$, where $\theta_t = \omega + \alpha \mathrm{r}_\mathrm{t}^2 + \beta h_{\mathrm{t}-1}$. Using the Bayes rule, it is seen that

$$\pi(h_t | h_{t-1}, f_t) \propto \pi(f_t | h_t) \pi(h_t | h_{t-1}, r_t).$$

Notice that

$$-\log\left(\pi(h_t|h_{t-1}, f_t)\right) \propto \frac{(BS^*(h_t) - f_t)^2}{\sigma_\varepsilon^2} + \frac{(h_t - \theta_t)^2}{\sigma_\zeta^2}.$$

By differentiating with respect to h_t, it is seen that the maximum a posteriori (MAP) estimate of h_t satisfies in the following updating equation

$$\Delta(h_t) + \frac{\sigma_\varepsilon^2}{\sigma_\zeta^2} h_t = f_t + \frac{\sigma_\varepsilon^2}{\sigma_\zeta^2} \theta_t,$$

where Δ is the delta Greek letter of financial derivative. The following proposition summaries the above discussion. Numerical methods say Newton-Raphson method may be applied to solve this equation.

Proposition 1. The MAP estimate of h_t satisfies in the following updating equation

$$\Delta(h_t) + \frac{\sigma_\varepsilon^2}{\sigma_\zeta^2} h_t = f_t + \frac{\sigma_\varepsilon^2}{\sigma_\zeta^2} \theta_t,$$

where Δ is the delta Greek letter of financial derivative.

Some Orders

In this section, it is shown that most of trading order strategies can be represented as functions $f(s, c, p, b)$ where c, p, s, b are call and put options, stock, and bond, respectively. A widely used type of $f(s, c, p, b)$ is the linear functions

$$f(s, b) = a_1 c + a_2 p + a_3 s + a_4 b,$$

Here, $a_i, i = 1,2,3,4$ are real numbers. For more details about trading orders see Nasdaq trader. (2014). In each strategy, m's are suitable thresholds defined in the order type.

a) Market order. A market order is an order to buy or sell a stock at the best available price. Generally, this type of order will be executed immediately. However, the price at which a market order will be executed is not guaranteed. It is important for investors to remember that the last-traded price is not necessarily the price at which a market order will be executed. In fast-moving markets, the price at which a market order will execute often deviates from the last-traded price or “real time” quotes. $f(s_T)$ of this type of order can be written as

$$f(s_T) = \min(s_T, m) = s_T - \max(s_T - m, 0).$$

That is, this strategy is a combination of a stock and call option on that specified stock.

b) Limit order. A limit order is an order to buy or sell a stock at a specific price or better. A buy limit order can only be executed at the limit price or lower, and a sell limit order can only

be executed at the limit price or higher. A limit order is not guaranteed to execute. A limit order can only be filled if the stock's market price reaches the limit price. While limit orders do not guarantee execution, they help ensure that an investor does not pay more than a predetermined price for a stock. Here, $f(s_T) = \max(s_T, m) = \max(s_T - m, 0) + m$.

c) Stop order. A stop order, also referred to as a stop-loss order, is an order to buy or sell a stock once the price of the stock reaches a specified price, known as the stop price. When the stop price is reached, a stop order becomes a market order. A buy stop order is entered at a stop price above the current market price. Investors generally use a buy stop order to limit a loss or to protect a profit on a stock that they have sold short. A sell stop order is entered at a stop price below the current market price. Investors generally use a sell stop order to limit a loss or to protect a profit on a stock that they own. Here, $f(s_T) = \max(s_T, m)$.

d) Stop-limit order. A stop-limit order is an order to buy or sell a stock that combines the features of a stop order and a limit order. Once the stop price is reached, a stop-limit order becomes a limit order that will be executed at a specified price (or better). Te benefit of a stop-limit order is that the investor can control the price at which the order can be executed. In this case,

$$f(s_T) = \begin{cases} \min(s_T, m_2) & s_T > m_1 \\ m_1 & s_T \le m_1 \end{cases}$$

Let $g(s_T) = \min(s_T, m_2)$. Thus, this order can be considered as a stop order defined on $g(s_T)$.

e) Fill-or-kill order. Another common special order type is Fill-or-Kill (FOK) order. An FOK order is an order to buy or sell a stock that must be executed immediately in its entirety; otherwise, the entire order will be cancelled (i.e., no partial execution of the order is allowed). Here,

$$f(s_T) = \begin{cases} s_{max} & s_T = s_{max} \\ 0 & \text{otherwise} \end{cases}$$

f) Market if touched. An MIT (market-if-touched) is an order to buy (or sell) an asset below (or above) the market. This order is held in the system until the trigger price is touched, and is then submitted as a market order. Again, $f(s_T) = \max(s_T - m, 0) - m$.

Other DP applications

In this section, other applications of dynamic programming (DP) technique in trading problem are studied.

Hedging Strategy

Although, the main focus of paper is the finding of optimal closing time of a trading position. However, in this section, first, the optimal portion α of financial asset s which is contributed in trading by traders is found. Indeed, we want to find α to minimize $f - \alpha s$ in each time t, under the physical probability measure $P: (p_{phs}, 1 - p_{phs})$, where notation phs stands for the physical. Here, it is assumed that the trader is a risk neutral one and $V_0(x) = \log(x)$. Notice that

$$V_k(f - \alpha s\ ) = \min_{0 \le \alpha \le 1} \Big(p_{phs} V_{k-1}(f_u - \alpha s u) + (1 - p_{phs}) V_{k-1}(f_d - \alpha s d) \Big).$$

Assuming, $ud = 1$, it is seen that

$$\alpha = \frac{1}{s}\{p_{phs} u f_d + \left(1 - p_{phs}\right) d f_u\}.$$

Here, f_d, f_u are values of derivatives using upper and lower future values sd, su of future price, of financial asset, in the trading. The following proposition summarizes the above discussion.

Proposition 2. The optimal hedge ratio is given by

$$\alpha = \frac{1}{s}\{p_{phs} u f_d + \left(1 - p_{phs}\right) d f_u\},$$

where, f_d, f_u are values of derivatives using upper and lower future values sd, su of future price, of financial asset, in the trading under the physical probability measure $P: (p_{phs}, 1 - p_{phs})$.

MDP Modeling

In this section, the MDP modeling and corresponding solution is proposed in a given stock market. Markov decision processes model decision making in stochastic, sequential environments. The essence of the model is that a decision maker, or agent, inhabits an environment, which changes state randomly in response to action choices made by the decision maker. The state of the environment affects the immediate reward obtained by the agent, as well as the probabilities of future state transitions. The agent's objective is to select actions to maximize a long-term measure of total reward. This article describes MDPs, an example application, algorithms for finding optimal policies in MDPs, and useful extensions to the basic model (see Ross, 1982).

To this end, consider a specified stock s, which generates cash flow of gains $f(s_i, u_i)$, at time $i \ge 1$, where $u_i = \pi(s_i)$ and π is paying policy. The state equation is given by $s_{i+1} = g(s_i, u_i)$. The present value of stock is given by $E\sum_{i=0}^{\infty}\gamma^i f(s_i, u_i)$ where, $\gamma = 1/(1+r)$, is discounted factor and r is discounted rate. It is interested to maximize $E\sum_{i=0}^{\infty}\gamma^i f(s_i, u_i)$ with respect to policy π. This problem defines a dynamic programming problem defined by value function as a recursive equation

$$V(s_i) = max_{\pi}\{f(s_i, u_i) + \gamma V(g(s_i, u_i))\}.$$

The following proposition summarizes the above discussion.

Proposition 3. The optimal policy is given by the argmax of following value function

$$V(s_i) = max_{\pi}\{f(s_i, u_i) + \gamma V(g(s_i, u_i))\},$$

where $\gamma = 1/(1+r)$, is discounted factor and r is discounted rate.

Data Analysis

In this section, the above theoretical results are summarized in a practical algorithm for trading strategy orders. Then, the mentioned algorithm is applied in a real data set.

Algorithmic plan

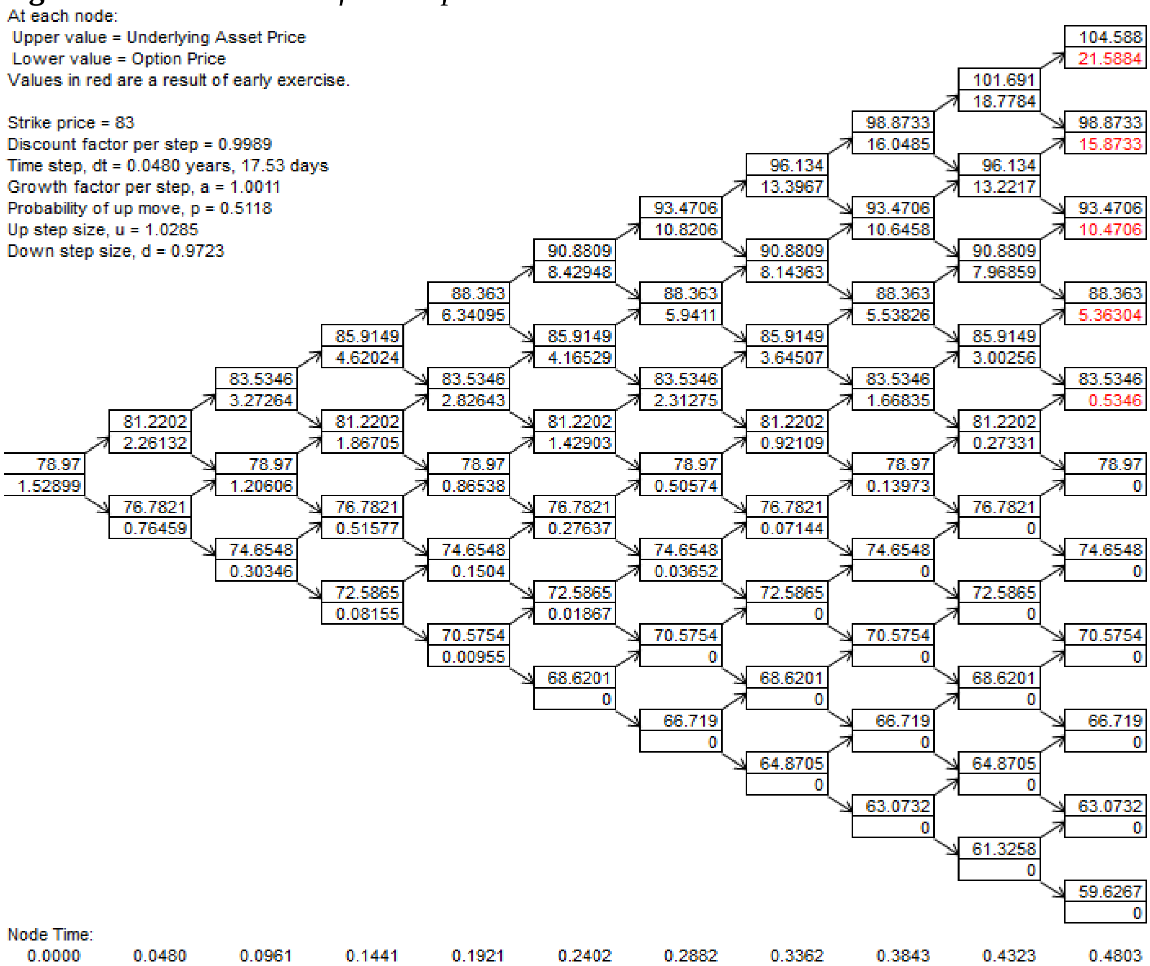
1. Derive the $f(s_T)$, for special strategy and compute the over-price that the trader should pay. Choose the minimum over-price order.
2. Assuming constant volatility, Using a binomial tree and based on dynamic programming in backward induction format, compute the optimal stopping as closing time of a specified trading position.
3. Assuming volatilities behave as a GARCH series and using the state space filtering technique, repeat the point 2.
4. Hedging strategies can be applied to remove the risk of a specified trading position. As well as, MDP techniques are applicable for finding the optimal dividend policy for policy makers as well as choosing the best stocks with optimal dividend policy for traders.

Next, consider a real data set containing 122 daily stock price of Apple Corporation for time period of 10 August 2017 to 2 February 2018. At the beginning (time zero), a trader buys one share of Apple Co. stock at the price $s_0 = 78.97$. The daily volatility estimate is $\sigma = 0.0080454$. Thus, the volatility per year is $\sqrt{254} \times 0.0080454 = 0.1282$. The mentioned trader considers a stop loss strategy with $m = 83$. The daily risk free rate is $\frac{2.2}{254}\%$. The maturity is $\frac{122}{254}$. Here, $f(s_T) = \max(s_T, m) = \max(s_T - m, 0) + m$.

Thus, using the binomial tree tool of *DerivaGem* software, the actual price of call option $\max(s_T - m, 0)$ at time zero is 1.5289. Also, the price of bond m

at time zero is $me^{-rT} = 83e^{-\frac{2.2}{100}\times\frac{122}{254}} = 82.127$. The fair price of this trading position is $82.127 + 1.5289 = 83.656$. Hence, the over-price paid for this strategy is -4.686 which produces an arbitrage opportunity. The binomial tree is plotted as follows.

Figure 1. *Binomial Tree of Call Option*



The optimal closing times of this strategy are only at the maturity. The delta of this trading position is 0.3372. So, it is enough to buy 0.3372 shares of stocks to delta neutral hedge.

Next, consider a GARCH(1,1) series for the volatility given by

$$\mathrm{h_t} = 0.00087107 + 1.06153194\mathrm{r}_{\mathrm{t}-1}^2,$$

where $r_t = \frac{s_t - s_{t-1}}{s_{t-1}}$. Here, the state space filter is applied. To derive σ_ε^2, the difference between empirical and theoretical (obtained using BS) prices of financial derivative is obtained. Then, the sample variance of these differences is an estimate of σ_ε^2 which is 5.52×10^{-9}. Indeed, m is chosen such that there is a call option for that maturity. Then, to estimate σ_ζ^2, sequential empirical estimates of volatilities are derived by $\frac{1}{t}\sum_{i=1}^{t} r_i^2$ and its differences between $h_t$obtained by a GARCH series produces ζ_t's. Then, their sample variance is an estimate of σ_ζ^2 which is 6.61×10^{-8}. Here, $\Delta(h_t) = \Phi(d_1)$, where Φ is the cumulative distribution function of standard normal distribution and $d_1 = \frac{\log\left(\frac{s}{m}\right)+\left(r+0.5v^2\right)(T-t)}{v\sqrt{T-t}}$. The MAP estimate of volatility and the fair price of stop-loss strategy are 1.84, 8.35, respectively, which considerably reduces the arbitrage opportunity.

Conclusions

Traders choose strategy to buy or sell at the maturity a financial asset such as stock. Indeed, they choose a financial derivative. Then, the fair price of is computable using Black-Scholes or binomial tree techniques. However, they pay the whole price of financial asset at the zero time. This over-price fee destroys the financial stability. Sometimes, it produces risk free return as an arbitrage opportunity. In this paper, this over-price fee is calculated and some hedging strategies are given. Beside this, using the Bayesian technique, the time varying volatility problems are solved.

References

Bjork, T. (2009). *Arbitrage theory in continuous time*. Oxford University Press. UK.

Liao, L. (2005). Stock option pricing using Bayes filters. *Technical Report*. Department of Computer Science. University of Washington. USA.

Nasdaq trader. (2014). Order types and modifiers. USA.

Ross, S. (1982). *Introduction to stochastic dynamic programming*. Academic Press. USA.

Shiryaev, A.N. and Novikov, A.A. (2008). On a stochastic version of the trading rule. ”Buy and Hold”. *Statistics and Decisions* 26, 289–302.

Shiryaev, A. N. and Zhitlukhin, M. V. (2013). Optimal stopping problems. *Technical report*. Steklov Mathematical Institute, Moscow and The University of Manchester, UK.

Tijms, H. (2012). Stochastic games and dynamic programming. *Asia Pacific Mathematics Newsletter* 3, 6-10.