

Quantum Souriau Lie Group Thermodynamics: A Comprehensive Review with Novel Insights and Results

By Mohamed El Morsalani

In 1969, Jean-Marie Souriau introduced "Lie Groups Thermodynamics" within the framework of Geometric Mechanics, offering a novel approach to Statistical Mechanics. F. Barbaresco and collaborators have demonstrated the applicability of Souriau's model across various domains, including information geometry and geometric deep learning. This paper presents a comprehensive review of the extension of Souriau's symplectic model to Quantum Information Theory. Building on the work of F. Barbaresco and F. Guy-Balmaz, who highlighted strong analogies between quantum information geometry and Lie group thermodynamics, this review explores the role of unitary representations of Lie algebras and the equivalence between the Fisher metric and the Bogoliubov-Kubo-Mori metric. In addition to the review, this paper introduces new results that further extend the classical Souriau framework by integrating modern developments in quantum thermodynamics. Specifically, this work links "Quantum Lie Groups Thermodynamics" to the geometry of coadjoint orbits, utilizing a geometric framework for mixed quantum states based on a Kähler structure. This framework, which encompasses a symplectic form, an almost complex structure, and a Riemannian metric, offers a comprehensive characterization of the space of mixed quantum states, providing deeper insights into the underlying geometric structure of quantum thermodynamics.

Introduction

J. M. Souriau developed a geometric theory of statistical mechanics, known as Lie Groups Thermodynamics, to address the issue that Gibbs equilibrium states do not conform to standard physical covariance principles. This theory is grounded in the Hamiltonian action of a Lie group on a symplectic manifold, where generalized Gibbs states are associated with a parameter β from the Lie algebra, which acts as a geometric analog to temperature (akin to Planck's temperature). In this framework, conventional Gibbs states, derived from a Hamiltonian, emerge as particular instances where the Lie group simplifies to a one-parameter group. For much of the material in this section I have been guided by the contents of (Barbaresco & Guy-Balmaz, 2020; Marle, 2020; Souriau, 1969).

Pioneering the Microscopic Theory of Matter

The origins of statistical mechanics can be traced back to Daniel Bernoulli (1700–1782), who, in his seminal work *Hydrodynamica* published in 1738, proposed a revolutionary idea: he conceived fluids (both gases and liquids) as composed of a vast number of moving particles. Bernoulli explained that the

1 pressure exerted by a fluid results from the collisions of these particles against the
2 walls of the container or any surface that measures the pressure. This concept laid
3 the groundwork for understanding the microscopic basis of macroscopic
4 phenomena, although it was not immediately recognized or widely accepted by the
5 scientific community of his time.

6 Bernoulli's insight into the particle nature of fluids remained largely
7 overlooked for more than a century. It wasn't until the mid-19th century that his
8 ideas gained traction, thanks to the work of several pioneering scientists. Rudolf
9 Clausius (1822–1888) was among the first to build on Bernoulli's concept,
10 beginning in 1857 to develop what became known as the kinetic theory of gases.
11 Clausius aimed to explain macroscopic properties of gases, such as temperature and
12 pressure, by deriving them from the equations governing the motion of individual
13 particles.

14 Around 1860, James Clerk Maxwell (1831–1879) made a significant
15 advancement by determining the probability distribution of particle velocities in a
16 gas at thermodynamic equilibrium, now known as the Maxwell-Boltzmann
17 distribution. Ludwig Boltzmann (1844–1906) extended this work further. In 1872,
18 he derived an evolution equation for the probability distribution of particle velocities
19 in gases that are not in equilibrium, now famously known as the Boltzmann
20 equation. By employing probabilistic arguments to describe particle collisions,
21 Boltzmann introduced a quantity denoted by H , which he showed always decreases
22 over time—a concept now recognized as a precursor to the second law of
23 thermodynamics. Boltzmann's H -theorem effectively relates to the concept of
24 entropy, with the H function being the negative of what we now understand as
25 entropy.

26 Building on these developments, Josiah Willard Gibbs (1839–1903)
27 formalized the principles of what would become a new branch of theoretical
28 physics—statistical mechanics. Gibbs' framework provided the tools to link
29 microscopic behavior with macroscopic observables, offering a profound
30 understanding of thermodynamics from a statistical perspective.

31 In the first half of the 20th century, it became clear that the motion of molecules
32 does not fully adhere to the laws of classical mechanics as initially assumed. Instead,
33 quantum mechanics was found to govern these microscopic systems. However, the
34 foundational concepts of statistical mechanics established by Gibbs proved to be
35 general and robust enough to be adapted to the quantum mechanical framework.
36 This adaptability allowed statistical mechanics to be applied not only to gases but
37 also to liquids and solids, thus broadening its scope and cementing its place as a
38 cornerstone of modern physics.

39 *Symplectic Souriau Lie Group Thermodynamics Model*

41
42 In this section we will introduce Souriau's Lie Group Thermodynamic model
43 in the symplectic case .

44
45

1 *Notations and Definitions*

2

3 For our setting we will need:

4

- 5 • A symplectic manifold (M, ω) is a manifold M endowed with a closed
- 6 nondegenerate valued 2-form where $\dim(M) = 2n$
- 7 • A symplectic Lie group action $\phi: G \times M \rightarrow M$ of G on M is symplectic,
- 8 meaning

9

$$\phi_g^* \omega = \omega, \quad \forall g \in G.$$

10

- The action is Hamiltonian and admits a symplectic momentum map

11

$$\mathfrak{J}: M \rightarrow \mathfrak{g}^*,$$

12

which satisfies

13

$$i_{\xi_M} \omega = d\mathfrak{J}_\xi,$$

14

where $\mathfrak{J}_\xi: M \rightarrow \mathbb{R}$ is defined by

15

$$\mathfrak{J}_\xi(m) = \mathfrak{J}(m) \cdot \xi = \langle J(m), \xi \rangle$$

16

and ξ_M is the infinitesimal generator of the action corresponding to $\xi \in \mathfrak{g}$

17

the Lie algebra of G and \mathfrak{g}^* the dual algebra.

18

- if M is connected, there is a group one-cocycle $\theta \in C^\infty(G, L(\mathfrak{g}, \mathbb{R}^n))$, θ

19

defined by

20

$$\theta(g) = \mathfrak{J}(\phi(m)) - Ad_{g^{-1}}^*(\mathfrak{J}(m))$$

21

where $Ad_{g^{-1}}^*$ is the coadjoint representation action.

22

- \mathfrak{g} the Lie algebra of G , whose elements will be denoted β the generalisations

23

of the inverse temperature.

24

- A duality pairing between elements ν of the dual space \mathfrak{g}^* and elements $\beta \in \mathfrak{g}$

25

denoted as $\langle \nu, \beta \rangle$.

26

- The manifold M is endowed with the Liouville volume form $d\mu$.

27

Denote by $\Omega \subset \mathfrak{g}$ the largest open set such that for all $\beta \in \Omega$ the two integrals

28

$$\int_M e^{-\langle \mathfrak{J}(m), \beta \rangle} d\mu \in \mathbb{R} \text{ and } \int_M \mathfrak{J}(m) e^{-\langle \mathfrak{J}(m), \beta \rangle} d\mu \in \mathfrak{g}^* \quad (1)$$

29

converge.

30

31

On Ω one can define the so-called partition function (or characteristic function)

32

$\psi: \Omega \rightarrow \mathbb{R}$, given by

33

$$\psi(\beta)$$

34

$$= \int_M e^{-\langle \mathfrak{J}(m), \beta \rangle} d\mu. \quad (2)$$

35

36

This permits us to define the generalized Gibbs probability densities associated

37

to each $\beta \in \Omega$:

38

$$p_\beta(m)$$

39

$$= \frac{1}{\psi(\beta)} e^{-\langle \mathfrak{J}(m), \beta \rangle} \quad (3)$$

40

41

1 **Remark 1.** *There are two facts which are very important to mention:*

- 2
- 3 • *For applications in information geometry it is required that $\beta \rightarrow p_\beta$ is*
- 4 *injective.*
- 5 • *The Gibbs densities are not defined on the whole vector space \mathfrak{g} but only on*
- 6 *the open subset Ω . From now on we assume that Ω is not empty. Its elements*
- 7 *are called geometric temperatures.*
- 8 • *If the integrals defining Ω are normally convergent then for example Ω is*
- 9 *convex.*

10

11 The generalized probability density ψ can be written using the Massieu

12 potential $\Phi: \Omega \rightarrow \mathbb{R}$

$$p_\beta(m) = e^{\Phi(\beta) - \langle \mathfrak{J}, \beta \rangle}, \quad \forall \beta$$

14 and

$$\begin{aligned} & \in \Omega & (4) \\ & \Phi(\beta) \\ & = -\log(\psi(\beta)) & (5) \end{aligned}$$

18

19 Another interesting quantity in this framework is the thermodynamic heat

20 $Q: \Omega \rightarrow \mathfrak{g}^*$ is the first derivative of the Massieu potential, i.e.,

$$Q(\beta) := D\Phi(\beta) = \int_M \mathfrak{J}(m) p_\beta(m) d\mu = \mathbb{E}_\beta[\mathfrak{J}] \in \mathfrak{g}^* \quad (6)$$

23

24 where \mathbb{E}_β denotes the expectation with respect to p_β .

25

26 *Entropy and some associated results*

27

28 Let Ω^* be the image of the Ω by the thermodynamic heat Q and assume that

29 $Q: \Omega \rightarrow \Omega^*$ is a diffeomorphism. In this case, the entropy $s: \Omega^* \rightarrow \mathbb{R}$ can be defined

30 as as the Legendre transform of the Massieu potential $\Phi: \Omega \rightarrow \mathbb{R}$, namely

$$s(v) := \langle v, \beta \rangle - \Phi(\beta), \quad \text{where } \beta$$

$$= Q^{-1}(v) \quad (7)$$

33 The next result has been proved in (Barbaresco & Gay-Balmaz, 2020).

34

35 **Lemma 2.** *For every $\beta \in \Omega$, the following equality holds true:*

$$s(Q(\beta)) = S(p_\beta), \quad \text{where } S(p) =$$

$$-\int_M p \log p d\mu. \quad (8)$$

38 S is the entropy of the probability density p and $Q(\beta)$ is the thermodynamic heat.

39

40 The following equivariance properties have been proved in a general

41 framework in (Barbaresco & Gay-Balmaz, 2020) and applied there also for the

42 symplectic case:

43

$$1 \quad (Ad_g)\Omega = \Omega, \quad \psi((Ad_g)\beta) = \psi(\beta)e^{\langle \theta(g^{-1}), \beta \rangle}, \quad p_\beta \circ \phi_g = p_{(Ad_{g^{-1}})\beta}$$

2 and

$$3 \quad \Phi((Ad_g)\beta) \\ 4 \quad = \Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle \quad (9)$$

$$5 \quad Q((Ad_g)\beta) \\ 6 \quad = (Ad_{g^{-1}}^*)(Q(\beta)) + \theta(g) \quad (10)$$

$$7 \quad s((Ad_{g^{-1}}^*)(v) + \theta(g)) \\ 8 \quad = s(v) \quad (11)$$

9 for every $g \in G$.

10

11 Furthermore Ω^* is invariant under the affine action $v \in \mathfrak{g}^* \mapsto (Ad_{g^{-1}}^*)(v) +$
12 $\theta(g) \in \mathfrak{g}^*$

13

14 *Fisher Metric and the maximum entropy principle*

15

16 The generalized heat capacity is the symmetric tensor field $K: \Omega \rightarrow \text{sym}(\mathfrak{g})$,
17 defined as minus the Hessian matrix of the Massieu potential, namely

$$18 \quad K(\beta) := -D^2\Phi(\beta) = D^2\log\psi(\beta): \mathfrak{g} \times \mathfrak{g} \\ 19 \quad \rightarrow \mathbb{R}. \quad (12)$$

20 As proved in (Barbaresco & Gay-Balmaz, 2020) the generalized heat capacity
21 has the following expression:

$$22 \quad K(\beta) = \mathbb{E}_\beta[(\mathfrak{I} - \mathbb{E}_\beta[\mathfrak{I}]) \otimes (\mathfrak{I} - \mathbb{E}_\beta[\mathfrak{I}])] \\ = \mathbb{E}_\beta[(\mathfrak{I} - Q(\beta)) \otimes (\mathfrak{I} - Q(\beta))]$$

23

24 As a consequence, $K(\beta)$ is positive semidefinite for all $\beta \in \Omega$. Being the
25 derivative of $Q: \Omega \rightarrow \Omega^*$ it is positive definite if Q is a diffeomorphism. As noticed
26 by F. Barbaresco in many of his papers there is a link between Souriau's work and
27 the information geometry that the authors apply in this general framework. Recall
28 that in information geometry, the Fisher metric associated with the family p_β , $\beta \in$
29 Ω of probability densities is the symmetric tensor field $I: \Omega \rightarrow \text{sym}(E)$ defined by

$$30 \quad I(\beta) \\ 31 \quad = -\mathbb{E}_\beta[D^2\log p_\beta]. \quad (13)$$

32

33 The authors in (Barbaresco & Gay-Balmaz, 2020) proved that in fact the
34 generalized heat capacity of p_β and the Fisher metric of p_β are equal:

35

$$36 \quad I(\beta) = K(\beta) = D^2\log\psi(\beta)$$

37

38 Furthermore they were able to show that if we assume that $Q: \Omega \rightarrow \Omega^*$ is a
39 diffeomorphism then the inverse of the Fisher metric, i.e., the cometric on Ω^* induced
40 from the Fisher metric on Ω , is given by minus the Hessian of the entropy:

$$-D^2s(v) =: E^* \times E^* \rightarrow \mathbb{R}, \quad \forall v \in \Omega^*.$$

There is also an equivariance property satisfied by the Fisher metric namely:

$$K\left((Ad_g)\beta\right)\left((Ad_g)(\delta\beta_1), (Ad_g)(\delta\beta_2)\right) = K(\beta)(\delta\beta_1, \delta\beta_2), \quad \forall g \in G. \quad (14)$$

One of the important results proved in the previous cited paper shows that the generalized Gibbs probability densities satisfy the maximum entropy principle.

Proposition 3. (*Maximum entropy principle*). Let $\mathfrak{S}: M \rightarrow \mathfrak{g}^*$ be a smooth function and $v \in \Omega^*$ a given element. The generalized Gibbs probability density p_β in Equation ([gibbs_density1]) with $\beta = Q^{-1}(v)$ is a solution of the maximum entropy

$$\text{principle: } \max_q \left[- \int_M q \log(q) d\mu \right] \text{ such that } \begin{cases} \int_M q d\mu = 1 \\ \int_M \mathfrak{S} q d\mu = v \end{cases}$$

Quantum Souriau Lie Group Thermodynamics model

In this section we apply the general framework (Barbaresco & Gay-Balmaz, 2020) for Gibbs probability densities in statistical mechanics and information geometry in the case of Unitary Representations and Quantum Fisher Metric as studied in the above cited paper.

Notations and definitions

For our setting we will need:

- Let G be a Lie group acting on a complex Hilbert space \mathcal{H} by a unitary left representation,

$$U_g: \mathcal{H} \rightarrow \mathcal{H}$$

- Let $\beta_{\mathcal{H}}$ be the associated infinitesimal generator, giving the Lie algebra representation and which is defined in the standard way by

$$\beta_{\mathcal{H}}(m) = \frac{d}{dt} U(\exp(t\beta))(m)|_{t=0} \quad \text{for } m \in \mathcal{H}, \quad (15)$$

and consider the auto-adjoint operator

$$-\beta_{\mathcal{H}}.$$

- We assume $\dim \mathcal{H} < \infty$. The following class of density matrices is considered

$$\rho_\beta = \frac{1}{\psi(\beta)} \exp(-\beta_{\mathcal{H}}) \quad (16)$$

for $\beta \in \mathfrak{g}$, with partition function

$$\begin{aligned} & \psi(\beta) \\ & = \text{Tr}(\exp(-\beta_{\mathcal{H}})) \end{aligned} \quad (17)$$

As remarked in (Barbaresco & Gay-Balmaz, 2020) the expression of the density matrices in ([density_matrices]) represent a generalization for the class of density matrices, which includes the class considered in (Nencka & Streater, 1999) and references therein.

Unitary representation of Lie groups

In the previous section we saw that F. barbaresco and F. Guy-Balmaz introduced the auto-adjoint operator $-\beta_{\mathcal{H}}$ in the density matrix ρ_{β} as a generalization of the unitary representations for the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$ studied in (Nencka & Streater, 1999). There was no mathematical justification for this choice which we will provide using arguments from the book of Kirillov (Kirillov, 2004) in the appendix V. titled representation theory.

The space \mathcal{H}^{∞} of smooth vectors

If \mathcal{U} is a unitary representation of Lie group G on \mathcal{H} then the matrix elements of $(\mathcal{U}, \mathcal{H})$ are bounded and continuous but not necessarily smooth functions especially if we are considering the infinite dimensional case.

Definition 4. A vector $m \in \mathcal{H}$ is called smooth if the vector-function $g \mapsto \mathcal{U}(g)m$ from G to \mathcal{H} is strongly infinitely differentiable. The set of all smooth vectors in \mathcal{H} is denoted by \mathcal{H}^{∞} .

As it was remarked by Kirillov in his book, the space \mathcal{H}^{∞} of smooth vectors is always dense in \mathcal{H} and it is big enough to reconstruct the representation $(\mathcal{U}, \mathcal{H})$ of a connected Lie group G from the representation of the Lie algebra \mathfrak{g} in \mathcal{H}^{∞} .

Self-adjoint infinitesimal generators of unitary presentations with domain \mathcal{H}^{∞}

The following theorem from (Kirillov, 2004) proves the existence of auto-adjoint infinitesimal generators of unitary presentations with domain \mathcal{H}^{∞} and thus justifying the generalization in the work of Barbaresco et al. in (Barbaresco & Gay-Balmaz, 2020):

Theorem 5. (Kirillov, 2004) Let $(\mathcal{U}, \mathcal{H})$ be a unitary representation of Lie group G . Then:

1. The subspace \mathcal{H}^{∞} of smooth vectors is dense in \mathcal{H} and stable with respect to all operators $\mathcal{U}(g)$, $g \in G$.
2. For any $\beta \in \mathfrak{g}$, the Lie algebra of G , the operator

$$A = -i\mathcal{U}_*(\beta) = -i \frac{d}{dt} \mathcal{U}(\exp(t\beta))|_{t=0}, \quad (18)$$

with the domain \mathcal{H}^{∞} is essentially auto-adjoint.

- 1 3. If G is connected, the representation $(\mathcal{U}, \mathcal{H})$ is completely determined by
 2 the representation \mathcal{U}_* of \mathfrak{g} in \mathcal{H}^∞ defined by (18). In particular, for any
 3 $m \in \mathcal{H}^\infty$ we have

$$4 \quad \mathcal{U}(\exp(t\beta))m =$$

$$5 \quad (\exp(itA))m \quad (19)$$

6 where the right-hand side is defined as the solution to the ordinary
 7 differential equation

$$8 \quad \frac{d}{dt}m(t) =$$

$$9 \quad \mathfrak{A}m(t) \quad (20)$$

10 with the initial condition $m(0) = m$.

11
 12
 13 As said earlier the proof of this theorem is from the book of Kirillov in the
 14 appendix V. about representation theory Theorem 4. In the following we will collect
 15 the preparatory steps for its demonstration as they are highly informative.

16
 17 A summary of the different steps of Kirillov's theorem proof

- 18 • For any $\beta \in \mathfrak{g}$ there correspond two vector fields on G : the left-invariant
 19 field R_β (infinitesimal right shift) and the right-invariant field L_β
 20 (infinitesimal left shift) given by

$$21 \quad (R_\beta f)(g)$$

$$22 \quad := \frac{d}{dt} f(g \exp(t\beta)) \Big|_{t=0}, \quad (L_\beta f)(g)$$

$$23 \quad := \frac{d}{dt} f(\exp(-t\beta)g) \Big|_{t=0}, \quad (21)$$

- 24 • For any function $\phi \in \mathcal{A}(G)$ define the operator $\mathcal{U}(\phi)$ on \mathcal{H} by the
 25 expression

$$26 \quad \mathcal{U}(\phi)$$

$$27 \quad = \int_G \phi(g) \mathcal{U}(g) d_l g \quad (22)$$

28 where $d_l g$ is a left-invariant measure on G .

- 29 • The following relation holds:

$$30 \quad \mathcal{U}_*(\beta) \mathcal{U}(\phi)$$

$$31 \quad = \mathcal{U}(L_\beta \phi) \quad (23)$$

- 32 • On \mathcal{H}^∞ the following relation holds:

$$33 \quad \mathcal{U}(\phi) \mathcal{U}_*(\beta)$$

$$34 \quad = -\mathcal{U} \left((R_\beta + \text{tr}(ad\beta)) \phi \right). \quad (24)$$

- 35 • From (23) it follows that for any $\phi \in \mathcal{A}(G)$ and any $m \in \mathcal{H}$ the vector
 36 $\mathcal{U}(\phi)m$ is smooth and therefore $\mathcal{U}(\phi)$ is called a smoothing operator.
 37 Kirillov then proves that the linear span of the images of all smoothing
 38 operators $\mathcal{U}(\phi)$, $\phi \in \mathcal{A}(G)$, is dense in \mathcal{H} .

- 39 • For $\beta \in \mathfrak{g}$ consider the operator $A = -\mathfrak{A}(\beta) = -\mathfrak{A}(\exp(t\beta)) \Big|_{t=0}$
 40 with \mathcal{H}^∞ as the domain of definition. Unitarity of $\mathcal{U}(g)$ leads to the fact

1 that A is symmetric: $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}^\infty$. To show that a
 2 symmetric operator A with domain D_A is essentially auto-adjoint, one uses
 3 the equivalence in the following criterion from Appendix IV.2.3 in (Kirillov,
 4 2004) :

$$5 \quad \text{Ker}(A^* + 1 \cdot \text{Id}) = \text{Ker}(A^* - 1 \cdot \text{Id}) = \{0\}.$$

- 6 • The relation (19) follows from the very definition of $\exp(itA)$.
- 7 • To reconstruct $(\mathcal{U}, \mathcal{H})$ from $(\mathcal{U}_*, \mathcal{H}^\infty)$ Kirillov argues in the following
 8 way. For any auto-adjoint operator \bar{A} the operator $\exp(it\bar{A})$ can be defined
 9 as follows. In the appropriate realization of \mathcal{H} in the form $L^2(\beta, \mu)$ the
 10 operator \bar{A} is just the multiplication by a real-valued function $a(x)$. Then
 11 one defines $\exp(it\bar{A})$ as a multiplication by $\exp(it a(x))$.
- 12 • When \bar{A} is the closure of A , this definition coincides with (18) on the
 13 subspace $D_A = \mathcal{H}^\infty$. Therefore $\mathcal{U}(\exp(t\beta))$ coincides with $\exp(itA)$ on
 14 \mathcal{H}^∞ and with $\exp(it\bar{A})$ on the whole space \mathcal{H} . Therefore, we know that the
 15 operators $\mathcal{U}(g)$ are in the neighborhood of the unit element covered by the
 16 exponential map. But, as G is connected, it is generated by any
 17 neighborhood of the unit.
- 18 • When $(\mathcal{U}, \mathcal{H})$ is a finite dimensional representation of a linear Lie group G
 19 then $\mathcal{H}^\infty = \mathcal{H}$. Hence \mathcal{U} is smooth as a mapping of G into $GL(\mathcal{H})$. In this
 20 paper we will be focusing on finite dimensional representations.

21 Properties of auto-adjoint infinitesimal generator of the unitary representation

22 Now using the proof's steps above let us transfer the exhibited properties there
 23 to the auto-adjoint infinitesimal generator of the unitary representation we will work
 24 with. Furthermore we will add some other known properties we will prove or cite
 25 their sources:
 26

$$27 \quad A(\beta) = -1\beta_{\mathcal{H}} = -1\mathcal{U}_*(\beta) =: -1\frac{d}{dt}\mathcal{U}(\exp(t\beta))|_{t=0}, \quad (25)$$

- 28 1. From the relation (19) we infer that for $m \in \mathcal{H}$:

$$29 \quad \mathcal{U}(\exp(t\beta))m = \exp(t\beta_{\mathcal{H}})m$$

30 (26)

- 31 2. The right-hand side of the previous equation is solution to the ordinary
 32 differential equation

$$33 \quad m'(t) = \beta_{\mathcal{H}}m(t) \quad (27)$$

34 with the initial condition $m(0) = m$.

- 35 3. For any function $\phi \in \mathcal{A}(G)$ and from the relations (23) and (24) we
 36 obtain:

$$37 \quad \begin{aligned} \beta_{\mathcal{H}}\mathcal{U}(\phi) &= \mathcal{U}(L_\beta\phi) \\ \mathcal{U}(\phi)\beta_{\mathcal{H}} &= -\mathcal{U}\left(\left(R_\beta + \text{Tr}(ad\beta)\right)\phi\right). \end{aligned}$$

38 (28)

- 39 4. The auto-adjoint infinitesimal generator $-1\beta_{\mathcal{H}}$ is symmetric:

$$\begin{aligned} & \langle -1\beta_{\mathcal{H}}x, y \rangle \\ & = \langle x, -1\beta_{\mathcal{H}}y \rangle. \end{aligned} \quad (29)$$

for all $x, y \in \mathcal{H}$.

5. Using the results from the book of Marsden and Ratiu (Marsden & Ratiu, 2003) (paragraph 9.3) the auto-adjoint infinitesimal generator $-1\beta_{\mathcal{H}}$ satisfies

$$\begin{aligned} & [-1\xi_{\mathcal{H}}, -1\eta_{\mathcal{H}}] \\ & = [\xi, \eta] \end{aligned} \quad (30)$$

6. Applying the adjoint presentation we have the following important property:

$$(Ad_g\beta)_{\mathcal{H}} = \mathcal{U}_g\beta_{\mathcal{H}}\mathcal{U}_g^{-1}$$

(31)

Indeed

$$\begin{aligned} (Ad_g\beta)_{\mathcal{H}} & = \frac{d}{dt}\mathcal{U}(\exp(tAd_g\beta))\Big|_{t=0} = \frac{d}{dt}\mathcal{U}(\exp(tg\beta g^{-1}))\Big|_{t=0} \\ & = \frac{d}{dt}\mathcal{U}(g)\mathcal{U}(\exp(t\beta))\mathcal{U}(g^{-1})\Big|_{t=0} \\ & = \mathcal{U}(g)\frac{d}{dt}\mathcal{U}(\exp(t\beta))\Big|_{t=0}\mathcal{U}(g^{-1}) \\ & = \mathcal{U}(g)\beta_{\mathcal{H}}\mathcal{U}(g^{-1}) \end{aligned}$$

7. We have another important property:

$$[\xi, \eta]_{\mathcal{H}} = [\xi_{\mathcal{H}}, \eta_{\mathcal{H}}] \quad (32)$$

Indeed, recall the known relation:

$$[\xi, \eta] = \frac{d}{dt}(e^{t\xi}\eta e^{-t\xi})\Big|_{t=0}$$

then

$$\begin{aligned} [\xi, \eta]_{\mathcal{H}} & = \left(\frac{d}{dt}(e^{t\xi}\eta e^{-t\xi})\Big|_{t=0}\right)_{\mathcal{H}} \\ & = \frac{d}{dt}\left(\left(e^{t\xi}\eta e^{-t\xi}\right)_{\mathcal{H}}\Big|_{t=0}\right) \text{ by linearity} \\ & = \frac{d}{dt}\left(\mathcal{U}(e^{t\xi})\eta_{\mathcal{H}}\mathcal{U}(e^{-t\xi})\Big|_{t=0}\right) \text{ by (31)} \\ & = \frac{d}{dt}\left(e^{t\xi_{\mathcal{H}}}\eta_{\mathcal{H}}e^{-t\xi_{\mathcal{H}}}\Big|_{t=0}\right) \text{ by (26)} \end{aligned}$$

8. The derivative of $\beta_{\mathcal{H}}$ with the respect to the inverse temperature β has the following expression:

$$\frac{d\beta_{\mathcal{H}}}{d\beta} = \beta_{\mathcal{H}}.$$

(33)

Indeed:

$$\begin{aligned} \frac{d\beta_{\mathcal{H}}}{d\beta} & = \frac{d}{d\beta}\left(\frac{d}{dt}\mathcal{U}(\exp(t\beta))\Big|_{t=0}\right) = \frac{d}{dt}\left(\frac{d}{d\beta}\mathcal{U}(\exp(t\beta))\Big|_{t=0}\right) \\ & = \frac{d}{dt}\left(t\mathcal{U}(\exp(t\beta))\Big|_{t=0}\right) = \frac{d}{dt}\left(\mathcal{U}(\exp(t\beta)) + t\mathcal{U}(\exp(t\beta))\Big|_{t=0}\right) \\ & = \beta_{\mathcal{H}} \end{aligned}$$

1 *Lie Groups Thermodynamics quantities*

2

3 Recall the expression of the density matrices ρ_β and the partition function we
4 will deal with

$$\begin{aligned} \rho_\beta &= \frac{1}{\psi(\beta)} \exp(-\mathbb{1}\beta_{\mathcal{H}}) \\ \psi(\beta) &= \text{Tr}(\exp(-\mathbb{1}\beta_{\mathcal{H}})) \end{aligned} \quad (34)$$

6

7 It is known in statistical physics that all thermodynamic properties can be
8 obtained from the partition function $\psi(\beta)$. Notice that

9

$$\text{Tr}(\rho_\beta) = 1.$$

11

12 In Souriau's Lie Groups Thermodynamics we need other important quantities.
13 The Massieu potential can be defined as follows:

14

$$:= -\log(\psi(\beta)) \quad \Phi(\beta) \quad (35)$$

16

17 Thermodynamic heat Q

18 Another interesting quantity in this framework is the thermodynamic heat Q
19 which is the first derivative of the Massieu potential, i.e.,

20

$$Q(\beta) := D\Phi(\beta) \quad (36)$$

22

23 The expression of the thermodynamic heat can be expressed using the
24 derivative of the partition function $\psi(\beta)$ as follows:

$$Q(\beta) = -\frac{d}{d\beta} \log(\psi(\beta)) = -\frac{\frac{d\psi(\beta)}{d\beta}}{\psi(\beta)}$$

26

27 Using the linearity of the trace function we have:

28

$$\begin{aligned} \frac{d\psi(\beta)}{d\beta} &= \frac{d}{d\beta} \left(\text{Tr}(\exp(-\mathbb{1}\beta_{\mathcal{H}})) \right) = \text{Tr} \left(\frac{d}{d\beta} (\exp(-\mathbb{1}\beta_{\mathcal{H}})) \right) \\ &= \text{Tr}(-\mathbb{1}\beta_{\mathcal{H}} \exp(-\mathbb{1}\beta_{\mathcal{H}})) \end{aligned} \quad (37)$$

30

31 and using the linearity of the trace function again and the linearity of the
32 infinitesimal generator (33) we obtain:

33

$$\begin{aligned} \frac{d^2\psi(\beta)}{d\beta^2} &= \frac{d}{d\beta} \text{Tr}(-\mathbb{1}\beta_{\mathcal{H}} \exp(-\mathbb{1}\beta_{\mathcal{H}})) \\ &= \text{Tr} \left(-(\beta_{\mathcal{H}}^2 + \mathbb{1}\beta_{\mathcal{H}}) \exp(-\mathbb{1}\beta_{\mathcal{H}}) \right) \end{aligned} \quad (38)$$

35

1 From the previous discussion we infer the following lemma:

2

3 **Lemma 6.** *The thermodynamic heat can be expressed using the derivative of the*
 4 *partition function $\psi(\beta)$ as follows:*

$$5 \quad Q(\beta) = -\frac{\frac{d\psi(\beta)}{d\beta}}{\psi(\beta)} =$$

$$6 \quad \frac{\text{Tr}(-1\beta_{\mathcal{H}} \exp(-1\beta_{\mathcal{H}}))}{\text{Tr}(\exp(-1\beta_{\mathcal{H}}))} \quad (39)$$

7 *and the first derivative of the heat can be expressed as follows:*

$$8 \quad \frac{dQ(\beta)}{d\beta} = \frac{\psi'{}^2(\beta)}{\psi^2(\beta)} - \frac{\psi''(\beta)}{\psi(\beta)}$$

$$= \frac{\text{Tr}^2(-1\beta_{\mathcal{H}} \exp(-1\beta_{\mathcal{H}}))}{\text{Tr}^2(\exp(-1\beta_{\mathcal{H}}))} - \frac{\text{Tr}(-(\beta_{\mathcal{H}}^2 + 1\beta_{\mathcal{H}}) \exp(-1\beta_{\mathcal{H}}))}{\text{Tr}(\exp(-1\beta_{\mathcal{H}}))} \quad (40)$$

9 **Definition 7.** *Recall that in quantum statistical systems using the concept of the*
 10 *density matrix then in equilibrium the expectation value of a physical observable is*
 11 *given by*

$$12 \quad \langle O \rangle_{\rho_{\beta}} = \text{Tr}(\rho_{\beta} O)$$

13 *Let ξ be an element in the Lie algebra \mathfrak{g} then the expectation value of 1ξ in the*
 14 *quantum state ρ_{β} is given as follows:*

$$15 \quad \langle 1\xi_{\mathcal{H}} \rangle_{\rho_{\beta}} = \text{Tr}(\rho_{\beta} 1\xi_{\mathcal{H}}) =$$

$$16 \quad \langle Q(\beta), 1\xi \rangle. \quad (41)$$

17 *Indeed:*

$$18 \quad \langle 1\xi_{\mathcal{H}} \rangle_{\rho_{\beta}} = \text{Tr}(\rho_{\beta} 1\xi_{\mathcal{H}})$$

$$= \text{Tr} \left(\rho_{\beta} 1 \frac{d}{dt} \mathcal{U}(\exp(t\xi)) \Big|_{t=0} \right)$$

19

20 The generalized heat capacity K

21 The generalized heat capacity is the symmetric tensor field K , defined as minus
 22 the Hessian matrix of the Massieu potential, namely

$$23 \quad K(\beta) := -D^2\Phi(\beta)$$

$$24 \quad = D^2 \log \psi(\beta). \quad (42)$$

25

26 The generalized heat capacity is computed as

$$27 \quad K(\beta)(\delta\beta_1, \delta\beta_2) = -D^2\Phi(\beta)(\delta\beta_1, \delta\beta_2)$$

$$= \text{Tr}(\rho_{\beta} 1(\delta\beta_1)_{\mathcal{H}} (\delta\beta_2)_{\mathcal{H}}) - \text{Tr}(\rho_{\beta} 1(\delta\beta_1)_{\mathcal{H}}) \text{Tr}(\rho_{\beta} 1(\delta\beta_2)_{\mathcal{H}})$$

28 (43)

29

30

31 thereby giving the covariance of the observables $1(\delta\beta_1)_{\mathcal{H}}$ and $1(\delta\beta_2)_{\mathcal{H}}$ in the
 32 quantum state ρ_{β} . In (Nencka & Streater, 1999), K is called the Bogoliubov-Kubo-
 33 Mori metric and chosen as the quantum version to the Fisher metric.

34

1 *Entropy and equivariance properties*

2

3 The von Neumann entropy s of the density matrix can be expressed in terms
4 of Massieu Potential Φ and the heat Q as follows

$$5 \quad -\text{Tr}(\rho_\beta \log \rho_\beta) = \text{Tr}(\rho_\beta \mathfrak{1}_{\mathcal{H}}) + \log(\psi(\beta)) = \langle Q(\beta), \beta \rangle - \Phi(\beta) \quad (44)$$

$$6 \quad = s(v),$$

6 for

7

$$v = Q(\beta) = \langle \mathfrak{1}(\cdot)_{\mathcal{H}} \rangle_\beta \in \mathfrak{g}^*$$

8

9 This is analogue to the result of Lemma 1 in (Barbaresco & Gay-Balmaz, 2020)
10 giving the entropy as the Legendre transform of $\Phi(\beta)$, thus giving a quantum
11 version of the Clairaut equation.

12 In the quantum case the adjoint representation is defined as follows:

$$13 \quad (Ad_g \beta)_{\mathcal{H}} = \mathcal{U}_g \beta_{\mathcal{H}} \mathcal{U}_g^{-1}. \quad (45)$$

14

15 Using $(Ad_g \beta)_{\mathcal{H}}$, we can infer the following equivariance properties, which
16 are obtained as in Proposition 5 (Barbaresco & Gay-Balmaz, 2020),

17

$$\psi(Ad_g \beta)_{\mathcal{H}} = \psi(\beta) \quad (46)$$

$$\Phi(Ad_g \beta)_{\mathcal{H}} = \Phi(\beta) \quad (47)$$

$$\rho_{(Ad_g \beta)_{\mathcal{H}}} = \mathcal{U}_g \circ \rho_\beta \circ \mathcal{U}_g^{-1} \quad (48)$$

18

$$Q(Ad_g \beta)_{\mathcal{H}} = (Ad_{g^{-1}}^* Q(\beta))_{\mathcal{H}} \quad (49)$$

$$s((Ad_{g^{-1}}^* v)_{\mathcal{H}}) = s(v) \quad (50)$$

$$K((Ad_g \beta)_{\mathcal{H}})((Ad_g \delta \beta_1)_{\mathcal{H}}, (Ad_g \delta \beta_2)_{\mathcal{H}}) = K(\beta)(\delta \beta_1, \delta \beta_2), \quad (51)$$

19 for every $g \in G$.

20

21 *The entropy as Casimir function and Kähler foliation of the dual algebra \mathfrak{g}^**

22

23 Let us define the Lie-Poisson bracket on \mathfrak{g}^*

24

$$\{f, g\}(\tau) = \left\langle \tau, \left[\frac{\delta f}{\delta \tau}, \frac{\delta g}{\delta \tau} \right] \right\rangle,$$

25

(52)

26

where \mathfrak{g} and \mathfrak{g}^* were identified using the duality pairing

27

$$\langle \tau, \beta \rangle = \text{Tr}(\tau^* \beta)$$

28

(53)

29

and view \mathfrak{g} as a Lie subalgebra of $\mathfrak{u}(\mathcal{H})$.

30

Theorem 8. *From the previous equivariance properties one can see that the
31 partition function ψ and Massieu potential Φ are constant on adjoint orbits.*

32

Furthermore the entropy s is a Casimir for the Lie-Poisson bracket on \mathfrak{g}^ which*

1 *foliates into the coadjoint orbits. Moreover the coadjoint orbits are the level sets of*
 2 *the entropy and possess a Kähler structure.*

3
 4 The last results generalize the one in (Nencka & Streater, 1999) where the
 5 authors studied the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$.

6 *Proof.* We will summarize the proof in the following bullet points:
 7

- 8 • From the equations (46) and (47) we see that the partition function ψ and
 9 Massieu potential Φ are constant on adjoint orbits.
- 10 • From the equation (50) one can infer easily that coadjoint orbits are the level
 11 sets of the entropy. Furthermore using the following result ((Laurent-
 12 Gengoux & Pichereau, 2013), Proposition 7.7) that says Casimir functions
 13 are the Ad^* -invariant functions, we can conclude the Casimir property of
 14 the entropy s .
- 15 • It is known that the symplectic leaves of a Lie-Poisson structure are exactly
 16 the coadjoint orbits of the coadjoint action when the Lie group G is
 17 connected. Therefore the dual-algebra \mathfrak{g}^* foliates into coadjoint orbits.
 18 Using the results discussed in the next section "The canonical Kähler
 19 structure for coadjoint orbits of compact Lie groups" of this article we can
 20 infer then the Kähler structure.

21 □

22 Furthermore this duality pairing implies that

$$23 \quad Ad_{g^{-1}}^* = Ad_g \quad \text{and} \quad ad_{\beta}^* \tau = [\tau, \beta].$$

24
 25
 26 This property yields to the following facts:

- 27 • Adjoint and coadjoint orbits are identified.
- 28 • The Kirillov-Kostant-Souriau symplectic form on coadjoint orbits has the
 29 followingh expression:

$$30 \quad \omega_{\mathcal{D}}(\tau)(ad_{\xi}\tau, ad_{\eta}\tau) = \langle \tau, [\xi, \eta] \rangle. \quad (54)$$

31
 32
 33 In (Barbaresco & Gay-Balmaz, 2020) the authors showed that Bogoliubov-
 34 Kubo-Mori metric on (co)adjoint orbits can be expressed as:

$$35 \quad \begin{aligned} K(\tau)(ad_{\xi}\tau, ad_{\eta}\tau) &= \langle ad_{\xi}^* Q(\tau), ad_{\eta}\tau \rangle \\ 36 \quad &= \langle Q(\tau), [\xi, [\eta, \tau]] \rangle \\ &= \langle \tau, [[Q(\tau), \xi], \eta] \rangle. \end{aligned} \quad (55)$$

37
 38 *Casimir dissipation/production*

39
 40 We recall here the general procedure as introduced in (Barbaresco & Gay-
 41 Balmaz, 2020). The general equations for Casimir dissipation/production Equation
 42 take into account the Lie algebra which is a subalgebra of $\mathfrak{u}(\mathcal{H})$ with the

1 identification $\mathfrak{g} = \mathfrak{g}^*$. Furthermore we consider a symmetric positive bilinear form
 2 given in this case by $\gamma(\beta, \tau) = \text{Tr}(\beta^* \tau)$. Let $\Lambda \neq 0$ be a parameter, $h: \mathfrak{g}^* \rightarrow \mathbb{R}$ be a
 3 Hamiltonian and $k: \mathfrak{g}^* \rightarrow \mathbb{R}$ be a function such that

$$4 \quad \left[\frac{\delta f}{\delta \tau}, \frac{\delta k}{\delta \tau} \right] \quad (56)$$

6 Then the general equations for Casimir dissipation/production are written as

$$9 \quad \frac{d}{dt} f = \{f, h\}(\tau) - \Lambda \left\langle \left[\frac{\delta f}{\delta \tau}, \frac{\delta k}{\delta \tau} \right], \left[\frac{\delta s}{\delta \tau}, \frac{\delta k}{\delta \tau} \right] \right\rangle \quad (57)$$

10 for every f , with $\{f, h\}$ the Lie-Poisson bracket Equation ([lie_poisson_bracket]).
 11 As $ad_{\beta}^* \tau = [\tau, \beta]$, using the results of Equation (41) in (Barbaresco & Gay-
 12 Balmaz, 2020) yield

$$15 \quad \frac{d}{dt} \tau + \left[\tau, \frac{\delta h}{\delta \tau} \right] = \Lambda \left[\left[\frac{\delta s}{\delta \tau}, \frac{\delta k}{\delta \tau} \right], \frac{\delta k}{\delta \tau} \right] \quad (58)$$

18 The canonical Kähler structure coadjoint orbits of compact Lie groups

19
 20 Coadjoint orbits appear in many areas of theoretical physics, for instance in
 21 representation theory, geometrical quantization, theory of magnetism, quantum
 22 optics. They serve as domains of definition in problems connected with nonlinear
 23 integrable equations (so called equations of soliton type). Since these equations have
 24 a wide application, the remarkable properties of coadjoint orbits interest not only
 25 mathematicians but also physicists. For much of the material in this section I have
 26 been guided by the contents of (Bordemann, Forger, & Römer, 1986) (Contreras,
 27 Ercolessi, & Schiavina, 2016) (Rieffel, 2009).

29 Complex and Kähler manifolds

30
 31 **Definition 9** (Complex Manifold). *A manifold Z that can be mapped on \mathbb{C}^n and
 32 with analytic diffeomorphism as compatibility condition between charts is called to
 33 be a Complex Manifold. Then, on the tangent bundle TZ we can define the complex
 34 structure $J: TZ \rightarrow TZ$ such that:*

$$36 \quad \forall v \in TZ: J(v) = -1 v. \quad (59)$$

37
 38 And $J^2 = -\text{Id}$.

39
 40 **Definition 10** (Kähler Manifold). *Let K be a real and even-dimensional and
 41 manifold with:*

42

- 1 • a complex structure J such that $J^2 = -\text{Id}$
 2 • a closed, non-degenerate two-form ω satisfying:

3
 4
$$\omega(x, Jy) + \omega(Jx, y) = 0 \quad (60)$$

5
 6 with $x, y \in TK$. In other words ω is a symplectic structure.

- 7 • a positive $(0,2)$ -tensor $g(\cdot, \cdot)$ such that:
 8 $g(\cdot, \cdot) =: \omega(\cdot, J(\cdot))$ meaning $g(x, y) =: \omega(x, Jy)$ (61)
 9 The equation (60) implies that g is symmetric and non-degenerate iff ω is
 10 non-degenerate. In this latter case g is a metric.

11
 12 In this case K is said Kähler Manifold.

13
 14 From the previous properties we can infer the following:

- 15 • The property $J^2 = -\text{Id}$ implies

16
 17
$$\omega(Jx, Jy) = \omega(x, y); \quad g(Jx, Jy) = g(x, y) \quad (62)$$

- 18 • The equation ([kaehler_metric_property]) yields:

19
 20
$$g(x, Jy) + g(Jx, y) = 0 \quad (63)$$

- 21 • Using equation (61) and substituting $y \mapsto Jy$ implies:

22
 23
$$\omega = -g(\cdot, J(\cdot)) \quad (64).$$

24
 25 This equation permits to define a Kähler manifold starting from a metric.
 26 Moreover, using the same tricks, we could start from g and ω and require that

27
$$J = g^{-1} \circ \omega$$

28
 29 is such that $J^2 = -\text{Id}$. In literature (J, g, ω) is said to be a compatible triple.

30
 31 *Coadjoint orbit at a distinguished point $\xi_\bullet \in \mathfrak{g}^*$*

32
 33 Let G be a connected compact semisimple Lie group, with Lie algebra \mathfrak{g} . Let
 34 \mathfrak{g}^* denote the dual Lie algebra.

35 Let us consider now a distinguished point $\xi_\bullet \in \mathfrak{g}^*$. The bullet denotes the
 36 representative role of ξ_\bullet in considering the class of all linear functionals in \mathfrak{g}^*
 37 obtained by acting on ξ_\bullet through the coadjoint-representation of the group G . As a
 38 matter of fact we are dealing with the orbit $\mathfrak{D}_{\xi_\bullet} \subset \mathfrak{g}^*$ defined by

39
$$\mathfrak{D}_{\xi_\bullet} = \{Ad_g^* \xi_\bullet \mid g \in G\}.$$

40
 41 Consider the subgroup H_{ξ_\bullet} of G that leaves ξ_\bullet fixed, defined by

42
$$H_{\xi_\bullet} = \{h \in G \mid Ad_h^* \xi_\bullet = \xi_\bullet\}.$$

43
 44 This subgroup is called stabiliser (isotropy) subgroup of (the point) ξ_\bullet . We have
 45 then

$$1 \quad \mathfrak{D}_{\xi_\bullet} \simeq G/H_{\xi_\bullet} \quad (65)$$

2

3 We denote $\mathfrak{h}_{\xi_\bullet}$ the Lie algebra of the stabiliser of ξ_\bullet . The action of G on $\mathfrak{D}_{\xi_\bullet} \subset$
 4 \mathfrak{g}^* is transitive and therefore it is possible to view the group G as a fibre bundle

5

$$6 \quad H_{\xi_\bullet} \rightarrow G \rightarrow \mathfrak{D}_{\xi_\bullet},$$

7

8 where the natural projection map, which is a G -equivariant diffeomorphism, is given
 9 by

$$10 \quad \begin{aligned} \pi: G &\rightarrow \mathfrak{D}_{\xi_\bullet} \\ g &\mapsto Ad_g^* \xi_\bullet \end{aligned}$$

11

12 with fibre above ξ_\bullet isomorphic to H_{ξ_\bullet} . By taking the tangent maps we obtain the
 13 exact sequence:

14

$$15 \quad 0 \rightarrow \mathfrak{h}_{\xi_\bullet} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{d\pi} T_{\xi_\bullet} \mathfrak{D}_{\xi_\bullet} \rightarrow 0 \quad (66)$$

16

17 where the map from \mathfrak{g} to $T_{\xi_\bullet} \mathfrak{D}_{\xi_\bullet}$ is given by the derivative $d\pi \equiv ad^*$ of the
 18 projection and $\iota: \mathfrak{h}_{\xi_\bullet} \hookrightarrow \mathfrak{g}$ is the inclusion of the stabilizing sub-algebra into \mathfrak{g} . A very
 19 important consequence of the exact sequence is the fact that

20

$$21 \quad \ker(ad^* \xi_\bullet) = \mathfrak{h}_{\xi_\bullet} \quad (67)$$

22

23 *A torus subgroup as subset in the center of H_{ξ_\bullet} .*

24

25 The killing form

26

Let us denote by \mathfrak{K} the negative of the Killing form of \mathfrak{g} :

27

$$28 \quad \mathfrak{K}(X, Y) = -Tr[ad_X ad_Y]. \quad (68)$$

29

30 Then \mathfrak{K} is positive-definite because G is compact. Using the Ad -invariance

$$31 \quad \mathfrak{K}(Ad_g(X), Ad_g(Y)) = \mathfrak{K}(X, Y), \quad X, Y \in \mathfrak{g}, g \in G$$

32

33 we can infer that the action Ad of G on \mathfrak{g} is by orthogonal operators with respect to
 34 \mathfrak{K} , and thanks to the ad -invariance

35

$$36 \quad \mathfrak{K}(ad_X Y, Z) = -\mathfrak{K}(Y, ad_X Z), \quad X, Y, Z \in \mathfrak{g}$$

37

38 we obtain that the action ad of \mathfrak{g} on \mathfrak{g} is by skew-adjoint operators with respect to
 39 \mathfrak{K} .

40

41 Because \mathfrak{K} is definite, it is a well known linear algebra result that for $\xi_\bullet \in \mathfrak{g}^*$
 42 there is an $\eta_\bullet \in \mathfrak{g}$ such that

$$\xi_{\bullet}(X) = \mathfrak{K}(X, \eta_{\bullet}) \text{ for all } X \in \mathfrak{g} \quad (69)$$

Furthermore the Ad -stability (isotropy) subgroup of η_{\bullet} namely

$$H_{\eta_{\bullet}} = \{h \in G \mid Ad_h \eta_{\bullet} = \eta_{\bullet}\}$$

Satisfies

$$H_{\xi_{\bullet}} = H_{\eta_{\bullet}} \quad (70)$$

Indeed, we have to prove that

$$\forall h \in G, Ad_h \eta_{\bullet} = \eta_{\bullet} \Leftrightarrow Ad_h^* \xi_{\bullet} = \xi_{\bullet}$$

We know that by definition

$$\langle Ad_h^* \xi_{\bullet}, X \rangle = \langle \xi_{\bullet}, Ad_{h^{-1}} X \rangle = \xi_{\bullet}(Ad_{h^{-1}} X) = \mathfrak{K}(Ad_{h^{-1}} X, \eta_{\bullet})$$

Using the fact that the action Ad of G on \mathfrak{g} is by orthogonal operators with respect to \mathfrak{K} the previous equality leads to:

$$\langle Ad_h^* \xi_{\bullet}, X \rangle = \mathfrak{K}(X, Ad_h \eta_{\bullet})$$

which yields the desired conclusion.

The torus subgroup T_{\bullet} .

A torus T is a compact connected abelian subgroup of G . A torus T is called maximal if there is no torus properly containing T . One can always obtain a torus subgroup by considering the subgroup $\exp(\mathbb{R}X)$ for $X \in \mathfrak{g}$. We apply these ideas here. Let T_{\bullet} be the closure in G of the one-parameter group

$$t \mapsto \exp(t\eta_{\bullet}), \quad t \in \mathbb{R}$$

so that T_{\bullet} is a torus subgroup of G .

In the following we are collecting some properties about T_{\bullet} :

- The stability subgroup $H_{\xi_{\bullet}}$ consists exactly of all the elements of G that commute with all the elements of T_{\bullet} .
- Note that the torus subgroup T_{\bullet} is contained in the center of $H_{\xi_{\bullet}}$ (but needs not coincide with the center).
- Since each element of $H_{\xi_{\bullet}}$ will lie in a torus subgroup of G that contains T_{\bullet} , it follows that $H_{\xi_{\bullet}}$ is the union of the tori that it contains, and so $H_{\xi_{\bullet}}$ is connected (Corollary 4.22 of). Thus for most purposes we can just work with the Lie algebra, $\mathfrak{h}_{\xi_{\bullet}}$, of $H_{\xi_{\bullet}}$ when convenient.
- In particular,

$$\mathfrak{h}_{\xi_{\bullet}} = \{X \in \mathfrak{g} \mid [X, \eta_{\bullet}] = 0\},$$

1 and $\mathfrak{h}_{\xi_\bullet}$ contains the Lie algebra, \mathfrak{t}_\bullet , of T_\bullet which results by passing to the
2 Lie algebras in the equality (70).

3

4 *Orthogonal of the toral Lie algebra*

5

6 Using the results in let $\mathfrak{m} = \mathfrak{h}_{\xi_\bullet}^\perp$ be the orthogonal subspace of $\mathfrak{h}_{\xi_\bullet}$ in \mathfrak{g} with
7 respect to the Killing form \mathfrak{K} . Since Ad preserves \mathfrak{K} , then \mathfrak{m} is carried into itself by
8 the restriction of Ad to H_{ξ_\bullet} . Thus

9

$$[\mathfrak{h}_{\xi_\bullet}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

10

11 It has been seen in the exact sequence (66), that \mathfrak{m} can be conveniently
12 identified with the tangent space to $T_{\xi_\bullet} \mathcal{D}_{\xi_\bullet}$ at the coset H_{ξ_\bullet} (which corresponds to
13 the point ξ_\bullet of the coadjoint orbit). In the next section a Kähler structure will be
14 defined for \mathfrak{m} .

15

16 *Complex structure on \mathfrak{m}*

17

18 The first component of the Kähler structure we will need is a symplectic form
19 ω_\bullet . This is the Kirillov- Kostant-Souriau form, defined initially on \mathfrak{g} by

20

$$21 \quad \omega_\bullet(X, Y) = \xi_\bullet([X, Y]) = \mathfrak{K}([X, Y], \eta_\bullet) = \mathfrak{K}(Y, [\eta_\bullet, X]). \quad (71)$$

22

23 Because η_\bullet is in the center of $\mathfrak{h}_{\xi_\bullet}$, we see that

24

$$\text{if } X \in \mathfrak{h}_{\xi_\bullet} \text{ then } \omega_\bullet(X, Y) = 0 \quad \forall Y \in \mathfrak{g}.$$

25

26 Conversely,

27

$$\text{if } X \in \mathfrak{g} \text{ and if } \omega_\bullet(X, Y) = 0 \quad \forall Y \in \mathfrak{g} \\ \text{then, as } \mathfrak{K} \text{ is nondegenerate, } [X, \eta_\bullet] = 0, \text{ so that } X \in \mathfrak{h}_{\xi_\bullet}.$$

28

29 Thus ω_\bullet "lives" on \mathfrak{m} and is nondegenerate there. Because Ad preserves \mathfrak{K} and
30 H_{ξ_\bullet} stabilizes η_\bullet , it is easily seen that the restriction of Ad to H_{ξ_\bullet} preserves ω_\bullet , that
31 is,

32

$$33 \quad \omega_\bullet(Ad_g(X), Ad_g(Y)) = \omega_\bullet(X, Y), \quad \forall X, Y \in \mathfrak{m}, \quad \forall g \in H_{\xi_\bullet}.$$

34

35 As proposed in (Rieffel, 2009) we follow the proof of Proposition 12.3 of
36 (Cannas da Silva, 2001) in order to construct a complex structure on \mathfrak{m} . Thanks to
37 the nondegeneracy of \mathfrak{K} , there is a unique linear operator, Γ_\bullet , on \mathfrak{m} such that

38

$$39 \quad \omega_\bullet(X, Y) = \mathfrak{K}(\Gamma_\bullet X, Y), \quad \forall X, Y \in \mathfrak{m}.$$

40 (72)

41

42 From equation (71) one can infer that Γ_\bullet is ad_{η_\bullet} restricted to \mathfrak{m} and so Γ_\bullet is
43 skew-symmetric, that is,

$$\Gamma_{\bullet}^{\dagger} = -\Gamma_{\bullet}.$$

This fact has been also cited in the proof of Proposition 12.3 of (Cannas da Silva, 2001). Furthermore using the nondegeneracy of \mathfrak{K} we get that Γ_{\bullet} is invertible and as η_{\bullet} is in the center of $\mathfrak{h}_{\xi_{\bullet}}$, the Ad -action of $H_{\xi_{\bullet}}$ commutes with Γ_{\bullet} . Let us define the polar decomposition of Γ_{\bullet} as follows:

$$\Gamma_{\bullet} = |\Gamma_{\bullet}|J_{\bullet}. \quad (73)$$

Since Γ_{\bullet} is invertible, so are $|\Gamma_{\bullet}|$ and J_{\bullet} , which yields that J_{\bullet} is an orthogonal transformation with respect to the Killing form \mathfrak{K} . Because Γ_{\bullet} is skew-symmetric, so is J_{\bullet} , so that

$$J_{\bullet}^{-1} = J_{\bullet}^{\dagger} = -J_{\bullet},$$

and J_{\bullet} commutes with $|\Gamma_{\bullet}|$. In particular,

$$J_{\bullet}^2 = -\text{Id}, \quad (74)$$

where Id denotes the identity operator on \mathfrak{m} . This means exactly that J_{\bullet} is a complex structure on \mathfrak{m} , preserved by the Ad -action of $H_{\xi_{\bullet}}$.

Kähler structure definition and Ad -action of T_{\bullet} on \mathfrak{m}

The final piece of structure is a corresponding inner product, g_{\bullet} , on \mathfrak{m} , defined by

$$g_{\bullet}(X, Y) = \omega_{\bullet}(X, J_{\bullet}Y) = \mathfrak{K}(\Gamma_{\bullet}X, J_{\bullet}Y) = \mathfrak{K}(|\Gamma_{\bullet}|X, Y) \quad (75).$$

Clearly g_{\bullet} is positive-definite, and is preserved by the Ad -action of $H_{\xi_{\bullet}}$, but we still need a reasonably explicit expression for the Riemannian metric corresponding to g_{\bullet} .

Ad -action of T_{\bullet} on the lie algebra \mathfrak{m}

Let us start by examining the Ad -action of T_{\bullet} on the Lie algebra \mathfrak{m} . By means of J_{\bullet} we make the Lie algebra \mathfrak{m} into a \mathbb{C} -vector space, by defining $1X$ to be just $J_{\bullet}X$ for $X \in \mathfrak{m}$. When we view \mathfrak{m} as a \mathbb{C} -vector space in this way we will denote it by \mathfrak{m}_J . Since the Ad -action of $H_{\xi_{\bullet}}$ (and thus of T_{\bullet}) on \mathfrak{m} commutes with J_{\bullet} , this action respects the \mathbb{C} -vector space structure. Define a \mathbb{C} -sesquilinear inner product, $\mathfrak{K}_{\bullet}^{\mathbb{C}}$ on \mathfrak{m} by

$$\mathfrak{K}_{\bullet}^{\mathbb{C}}(X, Y) = \mathfrak{K}(X, Y) + 1\mathfrak{K}(J_{\bullet}X, Y). \quad (76)$$

It is linear in the second variable. The Ad -action of $H_{\xi_{\bullet}}$ on \mathfrak{m}_J is unitary for this inner product. The Ad -action of T_{\bullet} on \mathfrak{m}_J then decomposes into a direct sum of one-dimensional complex representations of T_{\bullet} , whose corresponding representations of \mathfrak{t}_{\bullet} are given by real-linear functions on \mathfrak{t}_{\bullet} whose values are pure-imaginary (the "weights" of the ad -action). We let Δ_{\bullet} be the set of real-valued linear

1 functionals α on \mathfrak{t}_\bullet such that $\mathbb{1}\alpha$ is a weight of the ad -action. It will be convenient
2 for us to set, for each real-linear real-valued functional α on \mathfrak{t}_\bullet ,

$$3 \quad \mathfrak{m}_\alpha = \{X \in \mathfrak{m}_{J_\bullet} \mid ad_\eta(X) = \mathbb{1}\alpha(\eta)X = \alpha(\eta)J_\bullet X \ \forall \eta \in \mathfrak{t}_\bullet\}. \quad (77)$$

7 This yields to:

$$8 \quad \mathfrak{m}_\alpha = \{0\} \text{ exactly when } \alpha \notin \Delta_\bullet.$$

10 For any $X \in \mathfrak{m}_\alpha$ and $Y \in \mathfrak{m}_{J_\bullet}$, one infers from equation (71) that

$$11 \quad \begin{aligned} g_\bullet(X, Y) &= \omega_\bullet(X, J_\bullet Y) \\ 12 \quad &= \mathfrak{K}([\eta_\bullet, X], J_\bullet Y) = \mathfrak{K}(\alpha(\eta_\bullet)J_\bullet X, J_\bullet Y) \\ &= \alpha(\eta_\bullet)\mathfrak{K}(X, Y). \end{aligned} \quad (78)$$

14 Thus for $\alpha \in \Delta_\bullet$ and $X \in \mathfrak{m}_\alpha$ with $X \neq 0$ we have

$$16 \quad 0 < g_\bullet(X, X) = \alpha(\eta_\bullet)\mathfrak{K}(X, X), \quad (79)$$

18 and therefore

$$19 \quad \alpha(\eta_\bullet) > 0.$$

21 The above discussion permitted M. A. Rieffel to obtain the following
22 proposition describing $|\Gamma_\bullet|$ (Rieffel, 2009):

24 **Proposition 11.** *For each $\alpha \in \Delta_\bullet$, the restriction of $|\Gamma_\bullet|$ to \mathfrak{m}_α is*

$$26 \quad \alpha(\eta_\bullet)\text{Id}_{\mathfrak{m}_\alpha},$$

28 where $\text{Id}_{\mathfrak{m}_\alpha}$ is the identity operator on \mathfrak{m}_α . In particular, $\alpha(\eta_\bullet) > 0$ and on \mathfrak{m}_α
29 we have

$$31 \quad g_\bullet = \alpha(\eta_\bullet)\mathfrak{K}.$$

33 If P_α denotes the orthogonal projection of \mathfrak{m} onto \mathfrak{m}_α , then

$$35 \quad |\Gamma_\bullet| = \sum_{\alpha \in \Delta_\bullet} \alpha(\eta_\bullet)P_\alpha.$$

37 This proposition shows how strongly dependent g_\bullet is on the choice of ξ_\bullet . In
38 contrast, different ξ_\bullet 's that give η_\bullet 's that generate the same group T_\bullet may have the
39 same subspaces \mathfrak{m}_α .

1 **References**

2

3 Barbaresco, F., & Gay-Balmaz, F. (2020). *Lie Group Cohomology and (Multi)Symplectic*
4 *Integrators: New Geometric Tools for Lie Group Machine Learning Based on Souriau*
5 *Geometric Statistical Mechanics*. Entropy, 22, 498.

6 Bordemann, M., Forger, M., & Römer, H. (1986). *Homogeneous Kahler Manifolds: Paving*
7 *the Way Towards New Supersymmetric Sigma Models*. . Commun. Math. Phys. 102,
8 605-647 .

9 Cannas da Silva, A. (2001). *Lectures on symplectic geometry*. Berlin: Lecture Notes in
10 Math., 1764, Springer.

11 Contreras, I., Ercolessi, E., & Schiavina, M. (2016). : *On the geometry of mixed states and*
12 *the fisher information tensor*. J. Math. Phys. 57, 062209 .

13 Kirillov, A. A. (2004). *Lectures on the orbit method*. American Mathematical Society
14 Graduate studies in mathematics 64.

15 Laurent-Gengoux, C., & Pichereau, A. V. (2013). *Poisson Structures*. Berlin/Heidelberg,
16 Germany: Grundlehren der Mathematischen Wissenschaften. Springer: .

17 Marle, C. (2020). *On Gibbs States of Mechanical Systems with Symmetries*. J. Geom.
18 Symmetry Phys. 57: 45-85.

19 Marsden, J., & Ratiu, T. (2003). *Introduction to Mechanics and Symmetry*. Berlin,
20 Germany: Springer.

21 Nencka, H., & Streater, R. (1999). *Information geometry for some Lie algebras*. Infin.
22 Dimens. Anal. Quantum Probab. Relat. Top. 2, 441–460.

23 Rieffel, M. A. (2009). *Dirac operators for coadjoint orbits of compact Lie groups*. . Münster
24 J. of Math. 2 , 265–298.

25 Souriau, J.-M. (1969). *Structure des Systèmes dynamiques*. Paris, France: Dunod.

26