Causal Approach to Quantum Mechanics

This article develops a causal approach to quantum mechanics. The wave function $\psi(r,t)$ does not provide, according to this approach, a complete description of a physical system. The causal approach assumes the existence of 'hidden variables'- certain properties of elementary particles that would allow a physical system to be described using deterministic laws. In the present work, the following results were obtained using this approach: (1) the Schrödinger equation is a necessary condition for the stability of the motion of the electron in a hydrogen atom; (2) the motion of the electron along Bohr's orbits is stable; (3) the electron's spin precesses as it moves along Bohr's orbits, and (4) the energy of the precessional motion is determined by the Rydberg formula for the atomic spectrum. A critical analysis of Bohm's model is also presented and the mistakes in this model are analyzed. A detailed mathematical justification and an extensive development of these ideas can be found in the book Causal Interpretation of Quantum Mechanics by Nina Sotina.

Keywords: *a causal approach to quantum mechanics; the Schrödinger equation; the electron's spin precesses; the mistakes of Bohm's model;*

Schrödinger's Derivation of His Famous Equation and Its Critical Analysis

In 1926, Erwin Schrödinger submitted an article to the journal "Annalen der Physik" in which he proposed a linear differential equation for the wave function $\boldsymbol{\psi}$

$$\frac{\hbar^2}{2m_e}\Delta\psi + \left(\varepsilon + \frac{e^2}{r}\right)\psi = 0 \qquad (1.1)$$

where e is the charge of an electron, m_e is the mass of the electron. The eigenvalues of the equation turned out to be the known Bohr energy levels. It is not widely known that Erwin Schrödinger derived the equation by applying the methods of classical mechanics to the motion of the electron in a hydrogen atom. His derivation, however, contained mistakes. Nevertheless, the Schrödinger equation correctly gave Bohr's energy levels and was therefore accepted as a postulate of quantum mechanics. We will show below how these mistakes can be corrected and will demonstrate the mathematical results that follow from the classical mechanic's approach.

Let us briefly recall Schrödinger's original derivation. Schrödinger viewed a particle as a wave packet, because of this he searched for a wave equation, the eigenvalues of which would give the known Bohr energy levels. He started with the Hamilton-Jacobi equation for the motion of an electron

$$\frac{(\nabla S)^2}{2m_e} - \frac{e^2}{r} = \varepsilon \qquad (1.2)$$

where ε is a constant having dimension of energy, S is Hamilton's principal function, and e^2/r is the Coulomb potential. Next, Schrödinger introduced a new function ψ in the form

$$\psi(r) = exp(S/\kappa) \qquad (1.3)$$

He substituted (1.3) into Eq. (1.2) and obtained the quadratic form in ψ and its derivatives

$$F\left(\psi, \psi'_x,\ \psi'_y, \psi'_z,\right) = (\nabla\psi)^2 - \frac{2m_e}{\kappa^2}\left(\varepsilon + \frac{e^2}{r}\right)\psi^2 = 0 \qquad (1.4)$$

Schrödinger next searched for a real function ψ that would give an extreme value to the integral of the quadratic form (1.4).

As is known from the calculus of variations, the necessary condition for the existence of max or min of the definite integral

$$J = \iiint F(\ \psi, \psi'_x,\ \psi'_y, \psi'_z, x, y, z)d\tau$$

is $\delta J = \int df\delta\psi\frac{\partial\psi}{\partial n} - \iiint\left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\psi'_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial\psi'_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial\psi'_z}\right) - \frac{\partial F}{\partial\psi}\right]\delta\psi d\tau = 0$ (1.5)

where $d\tau$ is an elementary volume and the integral is taken over the whole space, df is an element of the surface over which the integral is taken. In view of that, the associated variational problem for the quadratic form (1.4) is

$$\delta J = \delta\iiint\left[(\nabla\psi)^2 - \frac{2m_e}{\kappa^2}\left(\varepsilon + \frac{e^2}{r}\right)\psi^2\right]d\tau = 0 \qquad (1.6)$$

hence

$$\int df\delta\psi\frac{\partial\psi}{\partial n} - \iiint\left[\Delta\psi + \frac{2m_e}{\kappa^2}\left(\varepsilon + \frac{e^2}{r}\right)\psi\right]d\tau\ \delta\psi = 0 \qquad (1.7)$$

From where he obtained

$$\Delta\psi + \frac{2m_e}{\kappa^2}(\varepsilon + \frac{e^2}{r})\psi = 0 \qquad (1.8)$$

and the surface integral $\int df\delta\psi\frac{\partial\psi}{\partial n} = 0$ (1.9)

In February 1926, Schrödinger added a section to his article, wherein he changed the condition (1.9) to the normalization condition

$$\iiint\psi^2 d\tau = 1 \qquad (1.10)$$

Schrödinger showed that (1.6) has solutions if ε satisfies the formula

$$\varepsilon_n = -\frac{m_e e^4}{2\kappa^2 n^2} \qquad (1.11)$$

The Critical Analysis of Schrödinger's Derivation

(1) From Schrödinger's substitution (1.4) it follows that function ψ is real. Eq. (1.1), however, has complex solutions. Rather than substitution (1.4), the correct substitution should be

$$\boldsymbol{\psi} = \mathrm{A}\boldsymbol{exp}(\boldsymbol{iS}/\hbar) \qquad (1.12)$$

As is shown in [1] subsequent inaccuracies followed from this Schrödinger's mistake, the correction of which, however, does not change equation (1.8).

(2) Schrödinger did not explain the physical meaning of the variational principle (1.6) which he used in his derivation. In 1929, N.G. Chetaev, a well-known expert in the theory of stability, worked at the University of Göttingen and became familiar with Schrödinger's work. He suggested that Schrödinger's equation (1.1), under condition (1.9), selects from the solutions of the Hamilton-Jacobi equation (1.2) those that satisfy the condition of stability for the motion. Unfortunately, Chetaev published his article on this topic in Russian, and it is not widely known to the world scientific community.

Chetaev's Interpretation of the Schrödinger Equation from the Standpoint of Stability Theory

In general, there are two approaches to the study of stability: one is to study the stability of a non-perturbed motion with an addition of small perturbing forces, and the other is to study the stability of motion under variation of the initial coordinates and momenta with the same primary forces. Chetaev used the first approach in the further derivation. The second approach is based on the Poincaré variational equations.

Chetaev's principle of stability. *Of all possible motions of the mechanical system, we consider those for which the free constants in the complete integral of the Hamilton-Jacobi equation (Jacobi's integral of motion) have fixed values. We will refer to the collection of these motions as a packet. In the packet, the points in the configuration space move as a front,* $\boldsymbol{S = const}$*, in the sense that if points at any moment of time* $\boldsymbol{t = t_0}$ *are at the surface* $\boldsymbol{S = c_0}$*, then they remain on the surface* $\boldsymbol{S = c}$ *at any later moment of time.*

We assume that the influence of perturbing forces, $\boldsymbol{W}$*, on a packet at an arbitrary point is proportional to the density of trajectories* A^2 *at this point. For the packet consisting of stable trajectories, this influence must be minimal. That is, the action of the perturbing forces is less for the packets for which*

$$\iiint \boldsymbol{WA^2 d\tau} \equiv \iiint \boldsymbol{W\psi\psi^* d\tau} \Rightarrow \boldsymbol{m\ i\ n} \qquad (2.1)$$

The integral is taken over the entire configuration space. Note, that each packet corresponds to a specific wave function $\boldsymbol{\psi}$.

In his attempt to derive the Schrödinger equation, Chetaev followed the same approach as Schrödinger starting with the Hamilton-Jacobi equation

$$\frac{(\nabla S)^2}{2m} + U = \varepsilon \qquad (2.2)$$

Here U is the potential energy associated with the given forces. He added the energy associated with small perturbing forces, W, into Eq. (2.2)

$$\frac{(\nabla S)^2}{2m} + U + W = \varepsilon \qquad (2.3)$$

Then Chetaev substituted W in Eq. (2.1) with the expression for W obtained from Eq. (2.3) and, thus, obtained the following variational problem

$$\delta \iiint \left(\frac{(\nabla S)^2}{2m} + U - \varepsilon\right) \psi\psi^* d\tau = 0 \qquad (2.4)$$

Note that Chetaev used the correct substitution (1.12) for a function ψ. From $\psi^*(\vec{r}) = A(\vec{r})\, exp(-iS/\hbar)$, it follows that

$$\frac{i}{\hbar}\frac{\partial S}{\partial x} = \frac{1}{\psi}\frac{\partial \psi}{\partial x} - \frac{1}{A}\frac{\partial A}{\partial x} = -\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial x} + \frac{1}{A}\frac{\partial A}{\partial x}$$
$$\frac{i}{\hbar}\frac{\partial S}{\partial y} = \frac{1}{\psi}\frac{\partial \psi}{\partial y} - \frac{1}{A}\frac{\partial A}{\partial y} = -\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial y} + \frac{1}{A}\frac{\partial A}{\partial y}$$
$$\frac{i}{\hbar}\frac{\partial S}{\partial z} = \frac{1}{\psi}\frac{\partial \psi}{\partial z} - \frac{1}{A}\frac{\partial A}{\partial z} = -\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial z} + \frac{1}{A}\frac{\partial A}{\partial z}$$

Hence, $\frac{1}{\hbar^2}\cdot\left(\frac{\partial S}{\partial x}\right)^2 = \left(\frac{1}{\psi}\frac{\partial \psi}{\partial x} - \frac{1}{A}\frac{\partial A}{\partial x}\right)\left(-\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial x} + \frac{1}{A}\frac{\partial A}{\partial x}\right) = -\frac{1}{\psi\psi^*}\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x} + \frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2$

$$-\frac{1}{\hbar^2}\cdot\left(\frac{\partial S}{\partial y}\right)^2 = -\frac{1}{\psi\psi^*}\frac{\partial \psi}{\partial y}\frac{\partial \psi^*}{\partial y} + \frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2 \qquad (2.5)$$

$$-\frac{1}{\hbar^2}\cdot\left(\frac{\partial S}{\partial z}\right)^2 = -\frac{1}{\psi\psi^*}\frac{\partial \psi}{\partial z}\frac{\partial \psi^*}{\partial z} + \frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

In view of Eq. (2.5), the variational problem (2.4) takes the form

$$\delta \iiint \left[\frac{\hbar^2}{2m}\left(\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \psi^*}{\partial y} + \frac{\partial \psi}{\partial z}\frac{\partial \psi^*}{\partial z} - \left(\frac{\partial A}{\partial x}\right)^2 - \left(\frac{\partial A}{\partial y}\right)^2 - \left(\frac{\partial A}{\partial z}\right)^2\right) + (U-\varepsilon)\psi\psi^*\right] d\tau = 0 \qquad (2.6)$$

Chetaev then applied method (1.6) to the variational problem (2.6) and obtained

$$\iiint \left[\frac{\hbar^2}{2m}\left(\Delta\psi - \frac{\Delta A}{A}\psi\right) - (U-\varepsilon)\psi\right] d\tau\delta\psi^* = 0 \qquad (2.7)$$

under condition

$$\iiint \psi\psi^* d\tau = 1 \qquad (2.8)$$

From Eq. (2.7), it follows that

$$\frac{\hbar^2}{2m}\Delta\psi + (\varepsilon - U - \frac{\hbar^2}{2m}\frac{\Delta A}{A})\psi = 0 \qquad (2.9)$$

Thus, in solving the problems of the stability of motion described by the Hamilton-Jacobi equation (2.2), Chetaev did not obtain the exact Schrödinger equation as a necessary condition for stability. As we can see, Eq (2.9) is the Schrödinger equation (for an arbitrary potential $\boldsymbol{U}$) plus an extra term $\boldsymbol{U_Q}$

$$U_Q = -\frac{\hbar^2}{2m}\frac{\Delta A}{A} \qquad (2.10)$$

Later, in 1952, David Bohm, using the Schrödinger equation as the starting point, obtained *the modified Hamilton-Jacobi equation,* that is, the Hamilton-Jacobi equation with an additional term $\boldsymbol{U_Q}$. Bohm substituted $\boldsymbol{\psi}$ in the correct form (1.12) in the Schrödinger equation, separated the imaginary and the real parts and obtained two equations:

$$\frac{(\nabla S)^2}{2m} + U + U_Q = 0 \qquad (2.11)$$

$$2(\nabla A)(\nabla S) + A\Delta S = 0 \qquad (2.12)$$

Bohm named $\boldsymbol{U_Q}$ *the quantum potential,* and the first equation *the modified Hamilton-Jacobi equation*. Bohm interpreted $\boldsymbol{|A|^2}$ as the probability density of the particle's position in a statistical ensemble of the particle's trajectories. The second equation of the system (2.12) he interpreted as a continuity equation for the probability density.

The nature of the quantum potential is still a matter of debate. There are many interpretations of what it represents. Bohm himself described the quantum potential as carrying *active information*. Some researchers believe that the quantum potential emerges from deeper subquantum dynamics; other hypothesis regarding the nature of the quantum potential also exist. It is important to note, however, that all these hypotheses rely on the incorrect assumption that the quantum potential in the form (2.10) acts throughout all space. We should also not that the quantum potential $\boldsymbol{U_Q}$ was obtained from the Schrödinger equation which, within the causal approach, is the necessary condition for the stability of a particle moving under the influence of a given potential $\boldsymbol{U}$ and certain addi-

tional forces. These forces can be characterized by $\boldsymbol{U_Q}$ only on the trajectories that follow from the Schrödinger equation. Outside these trajectories, these forces are not defined.

Below, we will present our own interpretation of the quantum potential.

Derivation of the Schrödinger Equation from Classical Mechanics by Introducing a New Potential

In addition to the given potential energy U, we introduce a term Φ into the Hamilton-Jacobi equation

$$\frac{(\nabla S)^2}{2m} + U + \Phi = \varepsilon \qquad (3.1)$$

Here ε is a constant having the dimension of energy and m is the mass of the particle. For now, we introduce Φ purely formally without going deeper into the physical processes that this function describes.

Let us now analyze the motion described by Eq. (3.1) from the point of view of stability theory. Following Chetaev's method, we introduced potential energy W associated with small perturbing forces, while keeping the same initial conditions. In this case, Eq. (3.1) takes the form

$$\frac{(\nabla S)^2}{2m} + U + \Phi + W = \varepsilon \qquad (3.2)$$

Condition (2.1) in our case take the form:

$$\delta \iiint F d\tau \equiv \delta \iiint \left(\frac{(\nabla S)^2}{2m} + U + \Phi - \varepsilon\right) A^2 d\tau = 0 \qquad (3.4)$$

As is known from the calculus of variations, functions S and A yield an extremum of a definite integral if they satisfy the system of two Euler–Lagrange equations

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial S'_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial S'_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial S'_z}\right) - \frac{\partial F}{\partial S} = 0 \qquad (3.5)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial A'_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial A'_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial A'_z}\right) - \frac{\partial F}{\partial A} = 0 \qquad (3.6)$$

and also, condition (2.8).

Assume that Φ does not change under a variation of S. If we take into account the relationship

$$\frac{(\nabla S)^2}{2m}\,A^2 = \frac{A^2}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] \tag{3.7}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial S'_x}\right) = \frac{\partial}{\partial x}\left(\frac{A^2}{m}\frac{\partial S}{\partial x}\right) = \frac{2A}{m}\frac{\partial S}{\partial x}\frac{\partial A}{\partial x} + \frac{A^2}{m}\frac{\partial^2 S}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial S'_y}\right) = \frac{\partial}{\partial x}\left(\frac{A^2}{m}\frac{\partial S}{\partial y}\right) = \frac{2A}{m}\frac{\partial S}{\partial y}\frac{\partial A}{\partial y} + \frac{A^2}{m}\frac{\partial^2 S}{\partial y^2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial S'_z}\right) = \frac{\partial}{\partial z}\left(\frac{A^2}{m}\frac{\partial S}{\partial z}\right) = \frac{2A}{m}\frac{\partial S}{\partial z}\frac{\partial A}{\partial z} + \frac{A^2}{m}\frac{\partial^2 S}{\partial z^2}$$

we can write (3.5) in the form

$$2(\nabla A)(\nabla S) + A\Delta S = 0 \tag{3.8}$$

Note that Eq. (3.8) coincides with Eq. (2.12). Consider now the second condition of the extremum, Eq. (3.6). If the function Φ does not depend on derivatives $A'_x,\ A'_y,\ A'_z$ but depends on A itself, then Eq. (3. 6) takes the form

$$\frac{(\nabla S)^2}{2m} + U + \Phi_0 + \frac{A}{2}\frac{\partial \Phi}{\partial A} = \varepsilon \tag{3.9}$$

If on the trajectories obtained from the Schrödinger equation under the causal approach, we set

$$\Phi_0 = U_Q = -\frac{\hbar^2}{2m}\frac{\Delta A}{A} \tag{3.10}$$

and also

$$\partial\Phi/\partial A = 0 \tag{3.11}$$

then Eq. (3.9) takes the form

$$\frac{(\nabla S)^2}{2m} + U - \frac{\hbar^2}{2m}\frac{\Delta A}{A} = \varepsilon \tag{3.12}$$

Eq. (3.12) coincides exactly with *the modified Hamilton-Jacobi equation,* Eq. (2.11), obtained by Bohm. It is important to emphasize here that the function Φ coincides with *the quantum potential* U_Q only on the trajectories determined by the Schrödinger equation. Outside these trajectories the function Φ is not known. This conclusion is fundamentally different from Bohm's erroneous as-

sumption that *the quantum potential* U_Q is defined on the *entire* configuration space.

As previously stated, Eq. (3.8) and Eq. (3.12) were obtained by Bohm from the Schrödinger equation by applying substitution (1.12). We now perform the reverse procedure and obtain the Schrödinger equation from Eq. (3.8), Eq. (3.12) and Eq. (1.12). Thus, from Eq. (1.12) we have

$$\frac{\partial^2\psi}{\partial x^2} = Ae^{\frac{iS}{\hbar}}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{i}{\hbar}\left(\frac{\partial^2 S}{\partial x^2} + \frac{2}{A}\frac{\partial A}{\partial x}\frac{\partial S}{\partial x}\right)\right]$$

$$\frac{\partial^2\psi}{\partial y^2} = Ae^{\frac{iS}{\hbar}}\left[\frac{1}{A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{i}{\hbar}\left(\frac{\partial^2 S}{\partial y^2} + \frac{2}{A}\frac{\partial A}{\partial y}\frac{\partial S}{\partial y}\right)\right] \qquad (3.13)$$

$$\frac{\partial^2\psi}{\partial z^2} = Ae^{\frac{iS}{\hbar}}\left[\frac{1}{A}\frac{\partial^2 A}{\partial z^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial z}\right)^2 + \frac{i}{\hbar}\left(\frac{\partial^2 S}{\partial z^2} + \frac{2}{A}\frac{\partial A}{\partial z}\frac{\partial S}{\partial z}\right)\right]$$

Let us now add the above equations

$$\frac{\Delta\psi}{\psi} = \frac{\Delta A}{A} - \frac{1}{\hbar^2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{i}{\hbar}\left((\nabla S) + \frac{2}{A}(\nabla A \cdot \nabla S)\right) \qquad (3.14)$$

If we now take into account conditions (3.8) and (3.12), we obtain from Eq. (3.14) the Schrödinger equation

$$\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) + (\varepsilon - U)\psi(\vec{r}) = 0 \qquad (3.15)$$

A Hydrogen Atom. The Rydberg Formula. The Precessional Motion of the Electron's Spin on Bohr's Orbits

We now apply the above approach to a hydrogen atom. As is known, the solutions of the Schrödinger equation for a hydrogen atom are written in the form $\psi_{nlk}(r,\vartheta,\varphi) = A_{nlk}(r,\vartheta)\ exp(ik\varphi),\ n = 1,2,\ldots,\infty,\ l \le n-1,\ k = 0,\pm1,..,\pm l$ (4.1)

where r , ϑ , φ are spherical coordinates. The velocity of the electron in the hydrogen atom, in terms of the Hamilton principal function, S , is

$$\vec{V} = (1/m_e)\nabla S \qquad (4.2)$$

In spherical coordinates, ∇S has the form:

$$\nabla S = \frac{\partial S}{\partial r}\bar{\imath}_r + \frac{1}{r}\frac{\partial S}{\partial \vartheta}\bar{\imath}_\vartheta + \frac{1}{r\,sin\,\vartheta}\frac{\partial S}{\partial \varphi}\bar{\imath}_\varphi$$

If we compare the solution (4.1) and the substitution (1.12), we can see that the phase of the wave function in terms of S is

$$\frac{S}{\hbar} = k\varphi \qquad k = 0, \pm 1, \pm 2, \ldots, \pm l \qquad (4.3)$$

Since S is independent of r and ϑ ,that is, $\partial S/\partial r = 0$ and $\partial S/\partial \vartheta = 0$, the components of the velocity of the electron along the radius $\bar{\imath}_r$ and along $\bar{\imath}_\vartheta$ are zero. Therefore, the expression (4.2) takes the form

$$\vec{V}_k(\vec{r}) = \frac{\hbar\, k}{m_e r\, sin\,\vartheta}\vec{\imath}_\vartheta \qquad (4.4)$$

It follows from Eq. (4.4) that only two situations are possible: either the center of mass of the electron is at rest ($k = 0$), or the center of mass is moving in a circular orbit lying in a plane that is parallel to the xy-plane with the center of the orbit at the z axis. Among these circular orbits, there are unstable ones that do not satisfy conditions (3.11). Condition (3.11), in the case of a hydrogen atom, is equivalent to two conditions

$$\partial U_Q/\partial \vartheta = 0 \qquad (4.5)$$
$$\partial U_Q/\partial r = 0. \qquad (4.6)$$

Let us consider the first condition (4.5). From Eq. (3.12) and Eq. (4.2), we have

$$U_Q = -\frac{\hbar^2}{2m_e}\frac{\Delta A_{nlk}}{A_{nlk}} = \varepsilon_n + \frac{e^2}{r} - \frac{\hbar^2 k^2}{2m_e r^2 sin^2\vartheta} \qquad (4.7)$$

From Eq. (4.7) we see that $\partial U_Q/\partial \vartheta = 0$ when $cos\,\vartheta = 0$. In other words, the orbits for which $cos\,\vartheta \neq 0$ are unstable. Therefore, for stable orbits the following must be true (for $sin\,\vartheta = 1$)

$$\vec{V}_k = \frac{\nabla S}{m_e} = \frac{\hbar k}{m_e r}\bar{\imath}_\varphi \qquad (4.8)$$

We see that the causal approach to the motion of an electron in the hydrogen atom gives us that only circular orbits with the center at the nucleus are stable.

Now let us consider condition (4.6). We can prove that, out of all the circular orbits with the center at the nucleus, this condition precisely extracts Bohr orbits. Indeed, if we set $sin\,\vartheta = 1$ in Eq. (4.7), and take the derivative $\partial U_Q/\partial r$, we obtain

$$\partial U_Q/\partial r = -\frac{e^2}{r^2} + \frac{\hbar^2 k^2}{m_e r^3} = 0 \qquad (4.9)$$

The solution of Eq. (4.9) for r gives us

$$r_{B_k} = \hbar^2 k^2/m\ e^2 \qquad (4.10)$$

which are Bohr orbits.

Conditions (4.6) and (4.5) also means that there is no force associated with the potential U_Q on Bohr orbits. Thus, we see that the motion of the center of mass of the electron in a hydrogen atom on stable trajectories occurs under the action of Coulomb force only. In this case the Newton's second law takes the form

$$\frac{m_e {V_k}^2}{r_{B_k}} = \frac{e^2}{{r_{B_k}}^2} \qquad (4.11)$$

and the energy of the center of mass of the electron, ε_k, is

$$\frac{m_e {V_k}^2}{2} - \frac{e^2}{r_{B_k}} = \varepsilon_k \qquad (4.12)$$

Recall that $\varepsilon_k = -\frac{e^4 m_e}{2\hbar^2 k^2}$ is Bohr's energy on *kth* orbit.

On Bohr's orbits, Eq. (3.12) takes the form

$$\frac{m_e {V_k}^2}{2} - \frac{e^2}{r_{B_k}} + U_Q = \varepsilon_n \qquad (4.13)$$

If we compare expressions (4.12) and (4.13), we see that *the quantum potential* on Bohr orbits is the difference between Bohr energies

$$U_Q = \varepsilon_n - \varepsilon_к = -\frac{e^4 m_e}{2\hbar^2 n^2} + \frac{e^4 m_e}{2\hbar^2 k^2} \qquad (4.14)$$

Thus, we have proved that *the quantum potential* U_Q on Bohr orbits is determined by the Rydberg formula.

Let us verify Eq. (4.14). As follows from the solutions of the Schrödinger equation for a hydrogen atom (4.1) for $n = 3,\ k = 2,$ we have

$$A_{32\pm2} = \frac{r^2}{r_o^2 162\sqrt{\pi}} e^{-\frac{r}{3r_0}} sin^2\vartheta$$

where $r_0 = \hbar^2/m_e e^2$. On a Bohr orbit $sin\vartheta = 1$. For $k = 2,$ Eq. (4.10) gives

$$r_{B_2} = 4\hbar^2/m\ e^2$$

The quantum potential, in this case, is

$$U_Q = -\frac{\hbar^2 \Delta A_{322}}{2m_e A_{322}} = -\frac{2\hbar^2}{m_e {r_{B_2}}^2} + \frac{\hbar^2}{m_e r_o r_{B_2}} - \frac{\hbar^2}{18 m_e r_0^2} = -\frac{e^4 m_e}{2\hbar^2 9} + \frac{e^4 m_e}{2\hbar^2 4}$$

$$= \varepsilon_3 - \varepsilon_2$$

As we can see, U_Q equals the difference between Bohr energy levels 3 and 2.

The Nature of the Quantum Potential (Hypothesis)

Now we present our interpretation of the quantum potential. An electron within the nucleus possess both angular momentum and spin. From the perspective of the causal approach, this means that it is possible to describe the motion of an electron with a spin as the motion of a spinning top its orbit. In such motion, according to the classical mechanics, there should exist the generalized motion integral (the Jacobi's integral) which depends on variables corresponding to both the motion of the center of mass and the rotational motion. Recall that the generalized motion integral is not the total energy of the system since it does not include a part of the kinetic energy containing terms that linearly depend on generalized velocities. In the case of the small precessional angle of a fast-spinning top and the center of mass moves with a constant speed $V,$ the generalized motion integral along a trajectory takes the from

$$\frac{m_e V^2}{2} + U + \Phi = const$$

In our case, Φ is the potential energy of the interaction between the electron's spin with an external field that is responsible for the precessional motion. As it follows from the theory of a fast-spinning top, Φ is approximately equal to the product of the magnitude of the angular momentum of the spinning top (which, for an electron, is $\frac{\hbar}{2}$) and the angular frequency of the spin's precession $\dot{\gamma}$

$$\Phi \approx \dot{\gamma} \ \frac{\hbar}{2}$$

We now return to the law of conservation of energy Eq. (3.12) for a quantum particle. We see that Eq. (4.14) coincides with Eq. (3.12) when $\Phi = U_Q$. Thus, on stable trajectories, for the angular frequency $\dot{\gamma}$ of the spin's precession we have

$$\dot{\gamma} \ \frac{\hbar}{2} = U_Q = \varepsilon_n - \varepsilon_к$$

We can represent $\dot{\gamma}$ as $\dot{\gamma} = \omega_n - \omega_к$, where $\omega_к$ and ω_n are the angular frequencies of the spin's precession that corresponds to energy $\varepsilon_к$ and ε_n. Therefore,

$$\frac{e^4 m_e}{2\hbar^2 k^2} = \frac{\hbar}{2}\omega_к \ .$$

The angular frequency $\omega_к$ is related to the angular frequency ω_{orb} of the electron's orbital motion around the nucleus. Indeed,

$$\omega_{orb} = \frac{V_k}{r_{B_k}} = \frac{e^4 m_e}{\hbar^3 k^3}$$

and we have $k\omega_{orb} = \omega_к$. That is, there exists a direct relationship between the precessional motion of the electron's spin and the electron's orbital motion.

The physical meaning of the second angular frequency, ω_n, which corresponds to ε_n, is more mysterious. To shed light on this, we need to understand

the physical processes associated with potential Φ. In my opinion, Φ reflects an interaction between the physical vacuum and a quantum particle.

Conclusion

If we introduce a new potential, (W), that reflects hidden variables in the Hamilton–Jacobi equation, then, under the causal approach, it is shown, on the basis of the theory of stability, that the Schrödinger equation is the condition for the stability of the motion of a particle under a given potential. In the case of the hydrogen atom, it can be shown that the electron's spin undergoes precession on the trajectories that satisfy the Schrödinger equation, and that the energy associated with this precessional motion is described by the Rydberg formula.

It should also be noted that the Bohm potential, in the form given by Eq. (3.10), is valid only on trajectories that satisfy the Schrödinger equation. Therefore, nonlocality is not required within our approach.

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