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Properties of 3-Triangulations for *p*-Toroid

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In this paper, a method for constructing a toroid and its decomposition into convex pieces is considered. A graph of connection for 3-triangulable toroid is introduced in such a way that these pieces are represented by graph nodes. It is shown that connected, nonorientable graph can serve as a graph of connection for some of the toroids. The relationship between graphs that can be realized on surfaces of different genus and corresponding toroids is considered.

Keywords: *3-triangulation of polyhedra, toroids, piecewise convex polyhedra, graph of connection*

Introduction

Polyhedron and *d*-dimensional polytope are generalizations of the term polygon in 3-dimensional and *d*-dimensional space. We can also generalize the process of triangulation into higher dimensions. Originally, triangulation is dividing a polygon with n vertices by $n-3$ diagonals into $n-2$ triangles and this can always be done. We shall use the same term - triangulation for its generalization in higher dimensions, or more specifically 3-triangulation, *d*-triangulation. In these cases, using only the original vertices, for 3-triangulation we divide the polyhedron into tetrahedra and for *d*-triangulation the *d*-polytope into *d*-simplices. Triangulation problems especially in 2- and 3-dimensional space and other types of polyhedron decomposition have significant applications in engineering and other fields of research (Zhang et al. 2018, Zhang et al. 2020).

But even for triangulation in 3-dimensional space, two new problems arise. The first is that it is not possible to triangulate certain non-convex polyhedra. One example is the famous Schönhardt's polyhedron (Schönhardt 1928). Another problem is, although it is possible to triangulate all convex polyhedra, different 3-triangulations of the same can have different numbers of tetrahedra (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005). This is the reason to consider the smallest (minimal) and the largest (maximal) number of tetrahedra in triangulation. It is shown that such values, linearly, resp. squarely depend on the number n of vertices.

Some properties of 3-triangulation for *p*-toroids will be considered, when triangulation is possible. A polyhedron topologically equivalent to a *p*-torus (i.e., sphere with p handles, $p \in N$ is a given natural number) is a *p*-toroid. The inspiration for this consideration was Szilassi (2005) definition of the torus-like polyhedron, which he called toroid. Here term "toroid" will be used as a common name for *p*-toroids for any $p \in N$, and Szilassi's toroid would be called 1-toroid.

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Under certain conditions, it is possible to 3-triangulate some toroids, although there are not convex. Examples of such 1-toroids are given in (Bokowski 2005, Császár 1949, Szilassi 1986, 2005, 2012); e.g., the Császár's polyhedron is 1-toroid with the smallest number of vertices. It has 7 vertices and it is triangulable with 7 tetrahedra. It was also discussed as a polyhedron without diagonals (Császár 1949, Szabó 1984, 2009). Additional examples of toroids are given in (Stojanović 2015, 2017, 2021, 2022) and some properties of their 3-triangulations are considered.

Here, after a brief overview of the previous results and the definitions of the necessary terms, a method for constructing a toroid based on a given graph as its graph of connection is given. Examples of *p*-toroids obtained in the introduced way will then be given with a discussion on the number *p* of the handles. Also for such toroids, the numbers of vertices and tetrahedra necessary for its 3-triangulation are calculated. These examples show that the number of tetrahedra is such that the lower limit given in the Theorem 1 (Stojanović 2022) is tight.

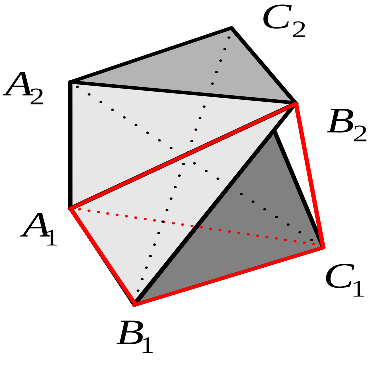
Preliminaries

The general properties of 3-triangulation of polyhedra are given first, and then properties for toroids. Necessary terms are introduced together with previously proven statements. Several examples are given to illustrate the introduced properties.

3-Triangulations of Simple Polyhedra

Although it is possible to triangulate all convex polyhedra, this is not the case with some of non-convex ones. A famous example of such a non-convex polyhedron was given by Schönhardt (1928) and shown in Figure 1. To obtain this polyhedron, we start with the trigonal prism $A_1B_1C_1A_2B_2C_2$ and triangulate its lateral faces with diagonals A_1B_2, B_1C_2 and C_1A_2. After that, we 'twist' the top basis $A_2B_2C_2$ for a small amount in the positive direction. Then none of the tetrahedra with vertices in the set $S = \{A_1, B_1, C_1, A_2, B_2, C_2\}$ would be inner. For example, the tetrahedron $A_1B_1C_1B_2$ has an edge C_1B_2 outside the Schönhardt's polyhedron. Tetrahedra $A_1C_1A_2B_2$ and $B_1C_1B_2C_2$ also contain the edge C_1B_2. For other tetrahedra with vertices in S, the situation is combinatorially the same as in some of the previous cases. Therefore, this polyhedron cannot be triangulated.

Figure 1. *Schönhardt Polyhedron*



Considering the smallest number of tetrahedra in the 3-triangulation of a polyhedron with n vertices we got that it is $n - 3$. E.g. such a polyhedron is pyramid V_{n-1} with $n - 1$ vertices in the basis and the apex, i.e., a total of n vertices. We can 3-triangulate it as follows: do any 2-triangulation of the basis into $(n - 1) - 2 = n - 3$ triangles. The apex together with each of such triangles makes one of the tetrahedra in 3-triangulation.

The triangular prism Π with bases $A_1B_1C_1$ and $A_2B_2C_2$ has 6 vertices and is also 3-triangulable with 3 tetrahedra. Actually, a triangular prism Π can be considered as a 'pyramid' with apex A_2 and spatial pentagon $A_1B_1B_2C_2C_1$ as the basis.

But not all polyhedra have the same property to have 3-triangulation with $n - 3$ tetrahedra. For example, bipyramids with $n - 2$ ($n \geq 5$) vertices in the basis and two apices can be triangulated in two different ways so that triangulations have respectively $2(n - 4)$ and $n - 2$ tetrahedra. Thus, a bipyramid with vertices A, B, C in the basis and apices V_1 and V_2 can be divided into two pyramids $ABCV_1$ and $ABCV_2$ in the first triangulation or into ABV_1V_2, BCV_1V_2 and CAV_1V_2 in the second. A special case of the bipyramid is the octahedron - a polyhedron with 6 vertices. It always gives 4 tetrahedra in 3-triangulation.

3-Triangulations that give small and especially minimal number ($T_{\min}$) of tetrahedra are examined in (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005).

Toroids and 3-Triangulation

We shall start with the term *p-torus*. In surface theory, it is defined as a cyclic polygon with paired sides. Any side s and its pair S are oppositely directed related to the fixed orientation of the polygon and then glued together. By a standard combinatorial procedure - the polygon can be divided and glued to a cyclic normal form $a_1b_1A_1B_1a_2b_2A_2B_2...a_pb_pA_pB_p$, as a p-torus. This combinatorial procedure is independent of the future spatial placement of the surface. So, from any spatial knot (as a topological circle in the space) we can form a p-torus. Of course, its surface can be 2-triangulated to be a surface of polyhedron.

Based on Szilassi's (1986) definition the term p-toroid is introduced (Stojanović 2021, Stojanović 2022).

Definition 1. *A polyhedron solid is called* p-toroid, $p \in N$ *, if it is topologically equivalent to a sphere with p handles* (*p-torus*).

As mentioned earlier, the term *toroid* will be used here as a common name for all p-toroids.

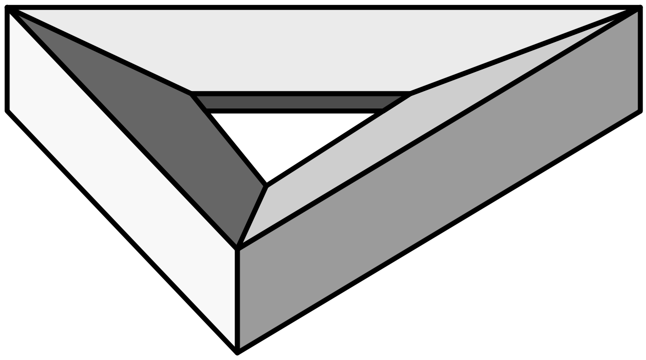
Császár's polyhedron is an example of a 1-toroid with the smallest number of vertices - 7. Its skeleton is the full graph with seven vertices that can be drawn on the torus, and so it has no diagonals. In Wolfram Demonstrations Project Szilassi

(Szilassi 2012) shows that Császár's polyhedron is 3-triangulable with 7 tetrahedra and it is a 1-toroid.

In (Szilassi 2005) Szilassi introduced *regular* 1-toroid. It is 1-toroid whose each face has *a* edges, and exactly *b* edges meet at each vertex. There are three classes of regular toroids, according to the number of edges incident with each face and each vertices. These classes are: T_1 where a=3, b=6, T_2 where a=4, b=4, T_3 where a=6, b=3.

The Császár's polyhedron is an example of a 1-toroid from class T_1. An example of a regular 1-toroid from class T_2 is given in Figure 2, marked here with P_9, since it has 9 vertices. We can see that the P_9 has 18 edges and 9 faces.

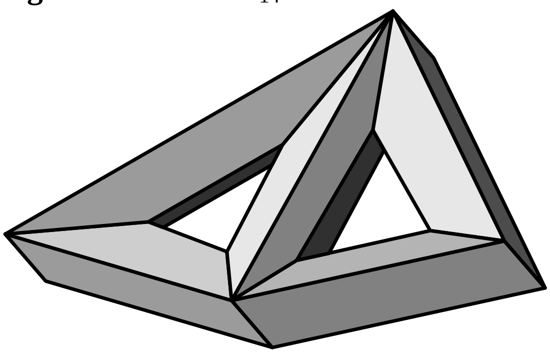
Figure 2. 1-*Toroid* P_9



Source: Stojanović 2015.

An example of 2-toroid P_{14} given in (Stojanović 2017) is shown in Figure 3. It consists of two glued P_9 and has 14 vertices, 32 edges and 16 faces.

Figure 3. 2-*Toroid* P_{14}



Source: Stojanović 2017.

Piecewise Convex Polyhedron and its Graph of Connection

Since toroids are not convex when considering their 3-triangulations, we shall use the following definitions.

Definition 2. *A polyhedron is* piecewise convex *if it can be divided into finitely many convex polyhedra P_i, $i = 1, \ldots, m$, with disjoint interiors. A pair of polyhedra P_i, P_j is said to be* neighbouring *if they have a common face called* contact face.

If the polyhedra P_i and P_j are not neighbouring, they may have a common edge e or a common vertex v. This is possible iff there is a sequence of neighbouring polyhedra $P_i, P_{i+1}, \ldots, P_{i+k} \equiv P_j$ such that the edge e, or the vertex v belongs to each contact face f_l common to P_l and P_{l+1}, $l \in \{i, \ldots, i+k-1\}$. Otherwise, the polyhedra P_i and P_j do not have common points.

Remark 1. *Since a convex polyhedron can be* 3*-triangulated, the same holds for a piecewise convex one, especially for a piecewise convex toroid.*
Remark 2. *Each* 3*-triangulable polyhedron is a collection of connected tetrahedra, so it is piecewise convex.*

In our investigation, we shall use the graph of connection for a piecewise convex polyhedron.

Definition 3. *If the polyhedron P is piecewise convex its* graph of connection (*or its* connection graph)*, is a graph with* nodes *representing convex polyhedra P_i, $i = 1, \ldots, m$, pieces of P, and* edges *representing contact faces between them.*

It is important to mention that the division of a polyhedron into convex pieces is not necessarily unique.

In order to have the same number of handles for the considered toroid P and the number of basic cycles of the corresponding connection graph, we shall introduce the term *optimized graph* of connection. Namely, it may happen that in the connection graph made as before, exists some 'false' cycle that do not correspond to some handle of P. Such a situation will disturb us. So, let us consider a toroid P and its graph of connection G that have one or more false cycles. Take all the nodes that belong to the same false cycle of G and the corresponding convex pieces of P. The union of such convex pieces builds a new node of the optimized graph G^*. In such a way we shall make new nodes for all the false cycles. The other nodes of the graph G remain in G^* and we shall call them the old ones. The set of edges for G^* consists of the previous edges between the old nodes, and the edges of G between some old node and some node belonging to a false cycle converted to the edge of G^* between that old node and the new one.

Note that it is not necessary for the new nodes of the optimized graph to correspond to convex polyhedra, they only correspond to simple piecewise convex polyhedra.

In earlier papers of the author (Stojanović 2015, Stojanović 2017, Stojanović 2022) there were proved the theorems for 1-toroids, 2-toroids and p-toroids about the minimal number of tetrahedra necessary for their 3-triangulation. Here we shall mention this one about p-toroids.

Theorem 1. *If a p-toroid with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is* $T_{\min} \geq n + 3(p-1)$.

In Stojanović (2021) it was considered how to construct a toroid *P* starting from a given graph *G*, in such a way that *G* would be a graph of connection for *P*. It is also shown that if for some simple polyhedron *S* the graph *G* is its skeleton, then the number *p* of the handles of the appropriate toroid *P* is the same as the number of faces *f* of *S* minus one, i.e. $p = f - 1$. Actually, to the question of whether the graph *G* has to be considered as "planar" or "spherical", answer is that it has to be "planar". Therefore, "outer face" surrounding these kinds of graphs should not be taken into account as some representing a handle.

Constructing *p*-Toroid from a Given Graph of Connection

Further properties and methods of constructing toroids based on the given graph will be considered here.

Theorem 2. *If the graph G is the skeleton of some polyhedron π (not necessarily simple) with f faces, then there exists a p-toroid* P' *whose optimized graph of connection is a subdivision of the graph G.*
Proof. For a given graph *G* which is the skeleton of some polyhedron *π*, we shall form a corresponding subdivided graph G' by splitting each edge of *G* and adding a new node between the splitted parts. Let us mark the old nodes in gray and the new ones in black.

Starting from the graph G', we shall form a toroid P' in such a way that each of the black nodes represents polyhedra of type *Π* and each of the gray nodes *v* of G' represents polyhedra of type V_k, where *k* is the number of edges from *v*. Since the nodes *v* of G' were originally vertices of the polyhedron *π*, $k \geq 3$ is always satisfied. Pieces of type *Π* and V_k can be connected in the following way: if $A_1, A_2, \ldots, A_k$ are vertices in the basis of V_k and *V* is the apex, then the contact faces of V_k, $k \geq 3$ and one of the bases of neighbouring polyhedra of type *Π* would be $A_i A_{i+1} V$, $i \in \{1, \ldots, k-1\}$, $A_k A_1 V$. If necessary, either the polyheda *Π* or V_k could be slightly deformed; especially polyheda *Π* could be with skew placed bases.

In this construction, the faces of the polyhedron *π* will be transformed into handles of P' because the inserted prisms *Π* allows the pyramids V_k to be far enough apart to form them. So, graph G' is an optimized graph of connection for toroid P'.

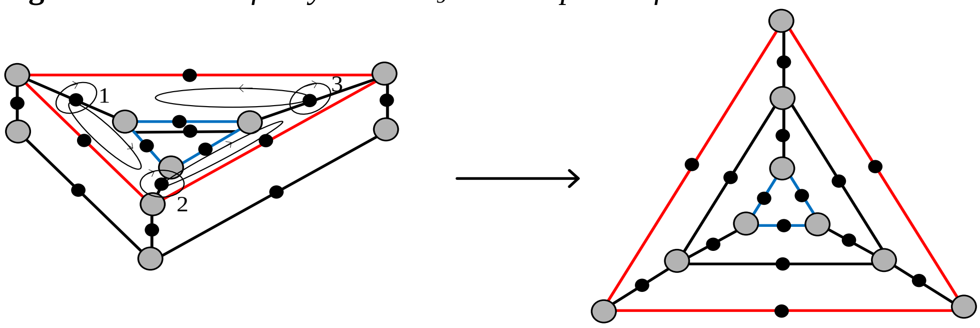
Note that the graph G' is visually similar to the toroid P', because the prisms *Π* looks like strings, and thus represent edges of P', while the pyramids V_k represent the vertices of P'. This is the reason why in the figures that illustrate the following examples we shall use the graph G' instead of the corresponding toroid P' and follow in parallel what happens to the mutually corresponding toroid P' and graph G'.

To calculate the number of tetrahedra in 3-triangulation, we mention that Π has 6 vertices and is 3-triangulable with 3 tetrahedra, while V_k has k+1 vertex and is 3-triangulable with $k - 2$ tetrahedra. Counting the vertices of P', it is sufficient to consider only the vertices of the pyramids of type V_k, because all the vertices of the prisms Π belong to some of the contact faces.

Example 1

We shall start with the 1-toroid P_9 given in Figure 2 as a polyhedron π. Let us denote with h the number of handles of the starting polyhedron. Since P_9 is a 1-toroid it holds h = 1. We shall then construct the corresponding graph G' and toroid P', as in the Theorem 2. For each of the 9 vertices of P_9, there are 4 edges that are incident to it, so we shall use V_4 as a component corresponding to the gray nodes of the graph G'. Since P_9 has18 edges, we shall use 18 prisms Π obtained form black nodes of G'.

Figure 4. *Skeleton of Polyhedron P_9 as a Graph G' of 10-Toroid*



Considering the number of handles for the obtained toroid P', we shall do more cutting in accordance with the definition of p-torus. The first 3 cutting are shown on the left side of Figure 4. After that, the only handle of P_9 will lead to two boundaries, i.e. two cycles of G'. The inside one is marked in blue and the outside in red. Then we have the situation as on the right side of Figure 4. We can see that it would be necessary 7 more cuttings to get a graph without a cycle or a corresponding polyhedron without handles. So, the constructed toroid P' has p = 10 handles.

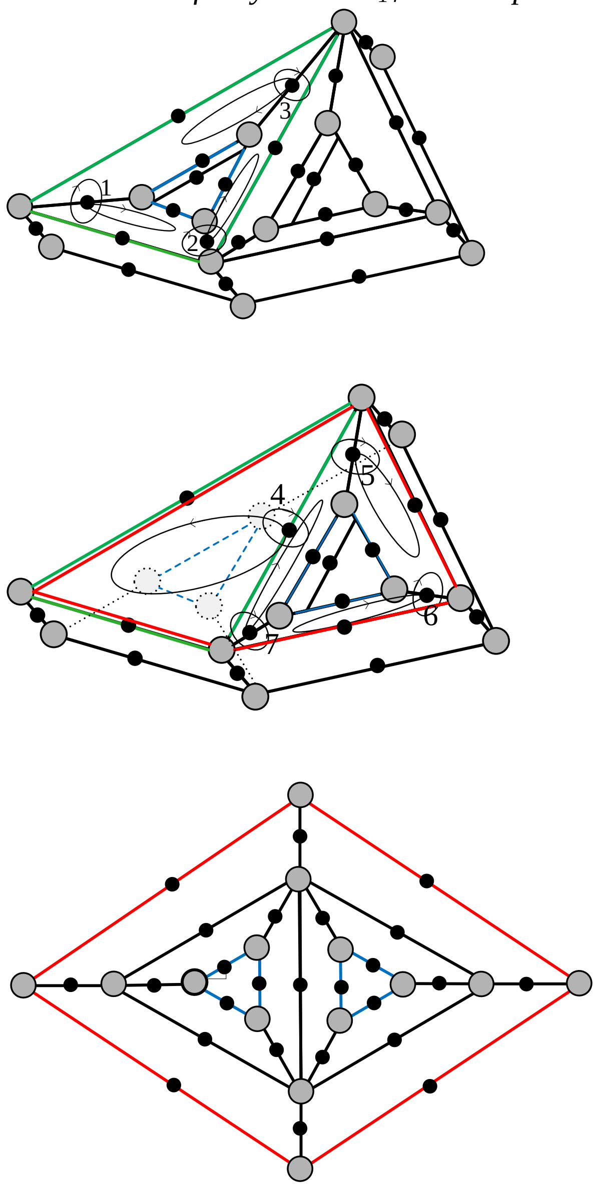
The number of vertices of P' is $n = 9 \cdot 5 = 45$, while the number of tetrahedra in the 3-triangulation is $T = 9 \cdot 2 + 18 \cdot 3 = 72$. Theorem 1 guarantees for P' that $T_{\min} \geq 45 + 3(10-1) = 72$. Thus, in this case, the lower limit given in Theorem 1 is reached.

We can conclude that the faces and the two boundaries obtained from the handle of P_9 lead to the handles of P' excluding "outer face". Since the number of faces of P_9 is f = 9, h = 1 and the number of handles of P' is 10, it holds $p = f + 2h - 1$.

Example 2

The 2-toroid P_{14} will be used as a polyhedron π, i.e. $h = 2$ in this example. For P_{14} it is also valid that it has 14 vertices, whereby 10 of them are incident to 4 edges and 4 of them are incident to 5 edges. So, the 10 components that correspond to the gray nodes of the graph G' will be pyramids V_4 and 4 of them V_5. There are 32 edges of P_{14} leading to 32 prisms Π. Note that for P_{14} the number of faces $f = 16$.

Figure 5. *Skeleton of Polyhedron P_{14} as a Graph G' of 19-Toroid*



In the cutting process, the first 3 of them are shown in the first part of Figure 5. As in the previous example, after this part of process, two boundaries will remain, marked in blue and green. The next 4 cutting are shown in the second part of Figure 5 with indicated previous green and dashed (invisible, behind green cycle) blue boundary. The green cycle gives the next cut and will not appear in the last part of process. It is also indicated that two blue ones remain - inner boundaries, the old and the new one, and also one red - outer boundary. In the end, there will remain a planar graph with 12 cycles (and one "outer", which is not taken into account). There are p = 19 handles of the new toroid P'. The number of handles p is also obtained by taking the number of faces f and two times the number of handles h of P_{14}. So, again is true $p = f + 2h - 1$.

Calculating the number of vertices of P' we obtain $n = 10 \cdot 5 + 4 \cdot 6 = 74$ and the number of tetrahedra in 3-triangulation is $T = 10 \cdot 2 + 4 \cdot 3 + 32 \cdot 3 = 128$. By Theorem 1, the lower limit for the number of tetrahedra is $T_{\min} \geq 74 + 3(19 - 1) = 128$ and it is reached again.

Conclusions

Using the concepts of piecewise convex polyhedra and of graph of connection, the properties of 3-triangulations for *p*-toroid, if any, are investigated. Based on the given graph as its connection graph, a *p*-toroid is constructed. Also, two examples of graphs and corresponding toroids are given. In the both cases, the considered graphs are skeletons of toroids. So, it was interesting to determine how many handles the toroid induced by the mentioned construction would have.

For *p*-toroids the minimal required number of tetrahedra for 3-triangulation is important property. That was the reason to examine also that number in the given examples. The result is that the lower limit obtained in the previous papers of the author has been reached.

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