Athens Journal of Sciences 2023, 10: 1-21
https://doi.org/10.30958/ajs.X-Y-Z

Some New Properties of Cyclotomic Cliques Arrangements

By Philippe Ryckelynck[*] & Laurent Smoch[±]

In this paper, we study the arrangement of lines in the euclidean plane constructed from the geometric clique graph generated by the regular n-gon. The vertices of this clique arrangement are located on finite concentric circles that we call orbits. We focus especially on the number of orbits and the number of vertices inside and outside the regular n-gon. Combinatorics in finite sets of quadruplets of integers provide information on the way the orbits are distributed. Next, using cyclotomic fields, we give galoisian properties of the radii of the orbits and their cardinalities.

Keywords: *arrangement of lines in the plane, cyclotomic fields, geometric graphs, Galois theory*

Introduction, Notation and Historical Notice

Let n be an integer and let $\mathcal{C}_n$ be the regular n-gon with vertices $z_m = \exp\left(\frac{2Im\pi}{n}\right)$, where $0 \leq m < n$ and $I = \sqrt{-1}$. If i, j, k, ℓ are four conveniently chosen indices then the two straight-lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ defined respectively by the two couples (z_i, z_j) and (z_k, z_ℓ) are secant and we denote $z_{i,j,k,\ell}$ as their intersection point. Let $\mathcal{K}_n$ the *geometric graph* which consists of the union of $\mathcal{C}_n$ together with the sets of straight lines $\mathcal{D}_{i,j}$ and intersection points $z_{i,j,k,\ell}$. We partition $\mathcal{K}_n$ in three sets as follows $\mathcal{K}_n = \mathcal{C}_n \cup \mathcal{K}_{e,n} \cup \mathcal{K}_{i,n}$, by defining respectively $z \in \mathcal{K}_{e,n}$ and $\mathcal{K}_{i,n}$ if and only if $|z| > 1$ and $|z| < 1$. The arrangement of lines in the plane which consists of the various straight lines $\mathcal{D}_{i,j}$, passing through the points z_i and z_j for all indices $i \neq j$ may be referred as the *clique arrangement* constructed from $\mathcal{C}_n$.

The aim of this paper is to state that all the intersection points $z_{i,j,k,\ell}$ are located on *circular orbits* centered at the origin, but also to characterize the radii of these orbits and to describe accurately their distribution. We will denote by $M_{e,n}$ and $M_{i,n}$ the respective numbers of exterior and interior orbits of the arrangement $\mathcal{K}_n$. We set $M_n = 1 + M_{e,n} + M_{i,n}$ which stands for the total number of orbits. Likewise, we will denote by $N_{e,n}$ and $N_{i,n}$ the respective cardinalities of the two sets $\mathcal{K}_{e,n}$ and $\mathcal{K}_{i,n}$.

Let $N_n = n + N_{e,n} + N_{i,n}$ be the whole number of intersection points.

---

[*]Associate Professor, Université du Littoral Côte d'Opale, France.
[±]Associate Professor, Université du Littoral Côte d'Opale, France.

Table 1. *Number of Intersection Points or Vertices of the clique*$\mathcal{K}_n$

n	3	4	5	6	7	8	9	10	11	12	13	14	…
N_n	3	5	15	37	91	145	333	471	891	901	1963	2185	…

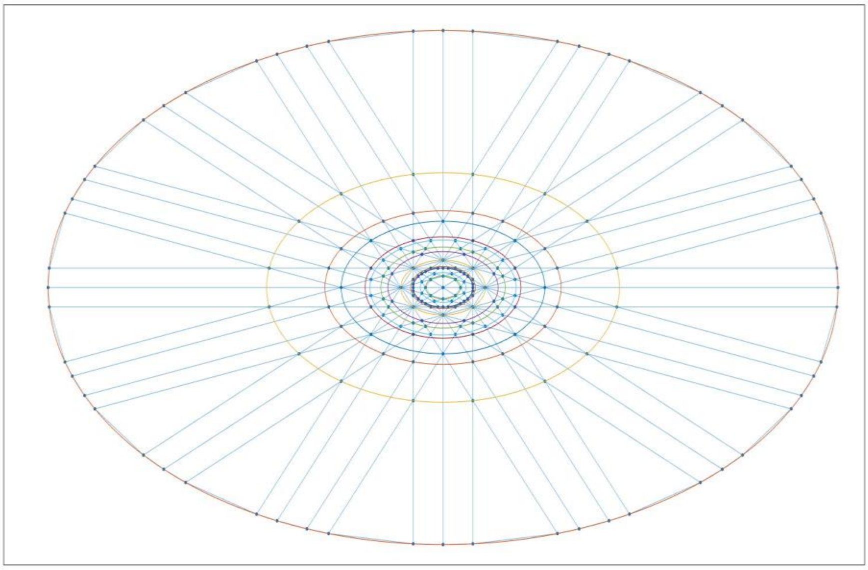
The sequence $(N_n)_{n\geq 3}$ is referred as A146212 in Sloane's (2023) OEIS. For odd n, it has been noted that $N_n = \frac{1}{8}n(n^3 - 7n^2 + 15n - 1)$ (see OEIS). We shall prove in Section 3 that these values N_n, which are not provided from a closed formula, may be however upper-bounded using inequalities on number lines by the sequence which starts as follows

Table 2. *Upper Bounds for Intersection Points of the clique* $\mathcal{K}_n$

n	3	4	5	6	7	8	9	10	11	12	13	14	…
q''_n	3	5	15	39	91	182	333	560	891	1347	1963	2765	…

Let us mention first that OEIS does not report this sequence, second that the upperbound is optimal for n even and at last, that the construction of the sequence (q''_n) is provided by a closed formula independent with respect to the parity of n.

Figure 1. *The Cyclotomic Clique Arrangement for* $n = 8$ *Together with its Circular Orbits*



Many researchers have been interested in describing and enumerating intersection points and polygons generated from an arrangement of lines and in particular from the diagonals of a regular n-gon. We may cite of course the results obtained by Bol (1936), Steinhaus (1958, 1983), Harborth (1969a, b), Tripp (1975), Rigby (1980) and especially those presented by Poonen and Rubinstein (1998), who were the first to provide two remarkable formulas for the number of *interior intersection points* made by the diagonals of a regular n-gon and the associated number of regions. Let us mention that the two authors provide also a brief but very interesting historical notice on the subject.

As an alternative of the formulas of Theorems 1 and 2 in Ryckelynck and Smoch (2023) related respectively to the number of interior points $N_{i,n}$ and the number of regions, one may devise two friendly written algorithms yielding those numbers. It is remarkable that the final outputs of these algorithms are polynomials on each residue class modulo 2520. Here are the first values of $N_{i,n}$

Table 3. *Sequence ($N_{i,n}$) A006561 in OEIS*

n	3	4	5	6	7	8	9	10	11	12	13	14	…
$N_{i,n}$	0	1	5	13	35	49	126	161	330	301	715	757	…

By substracting these values to the total number points of the graph and the points on the unit circle, i.e., by calculating $N_{e,n} = N_n - n - N_{i,n}$, we get the sequence of exterior points numbers of the cliques arrangements

Table 4. *Sequence ($N_{e,n}$) A146213 in OEIS*

n	3	4	5	6	7	8	9	10	11	12	13	14	…
$N_{e,n}$	0	0	5	18	49	88	198	300	550	588	1235	1414	…

This sequence is defined, for n odd given only, through the formula $N_{e,n} = n(2n^3 - 15^2 + 34n - 21)/24$. The authors also compute the number of regions formed by the diagonals, by using Euler's formula $V - E + F = 2$.

In the recent paper (Ryckelynck and Smoch 2023), we got interest in cyclotomic arrangements of lines in the plane. We initiated a new approach to the problem of enumerating convex connected components (known as chambers), depending on their shapes and their compacity. We presented, in an analogous problem, the description of each chamber of the *boundification* of the space defined as the complement of an arrangement of lines associated to the regular n-gon. Symmetry, circular orbits, goniometric functions are the core of this paper. Nevertheless, the whole program of work given in this work was too much difficult to replicate in the case of the clique arrangement $\mathcal{K}_n$ of the present paper, and we take as an objective to explain the way the intersection points are in fact "regularly" distributed along some concentric circles.

Let us present the notation used throughout this paper. First, $\sharp(S)$ denotes the cardinality of some finite set S, $\Im(f)$ denotes the range of the mapping $f: S_1 \to S_2$. As usual, if $x \in \mathbb{R}$ then $[x]$ denotes the greatest integer function. Let us use the abbreviation $pq_{\pm} = (p \pm q)\frac{\pi}{n}$ for convenient integer indices p and q. Although we may use indices modulo n, we impose instead the inequalities $0 \leq i \leq n - 1$ over all subsequent indices.

Next, $\varphi(n)$ denotes as usual the Euler's totient function. Let $\zeta_n = \exp\left(\frac{2\pi}{n} I\right)$ be a n-th primitive root of unity so that $\zeta_n^n = 1$, and let $\mathbb{Q}(\zeta_n)$ be the cyclotomic field generated by ζ_n.

For various integers n, i, j, k, ℓ, such that $i + j - (k + \ell)$ is not a multiple of n we set

$$J_n(i,j,k,\ell)=\frac{\cos^2(ij_-)+\cos^2(k\ell_-)-2\cos(ij_-)\cos(k\ell_-)\cos(ij_+-k\ell_+)}{\sin^2(ij_+-k\ell_+)}.$$

When $k+\ell\not\equiv 0 \bmod n$ and $i+j\equiv 0 \bmod n$, we set

$$\tilde{J}_n(i,j,k,\ell)=\frac{\cos^2(ii_+)+\cos^2(k\ell_-)-2\cos(ii_+)\cos(k\ell_-)\cos(k\ell_+)}{\sin^2(k\ell_+)}.$$

Let us consider the following finite set $\mathcal{R}_n=\Im(J_n)\cup\Im(\tilde{J}_n)$. The numbers in $\mathcal{R}_n$ are nothing but the squares of the radii of the orbits containing the vertices of $\mathcal{K}_n$. We have $\mathcal{R}_n\subset\left[0,\left(\sin\left(\frac{\pi}{n}\right)\right)^{-2}\right]$. Defining $\mathcal{R}_n$ shall allow to characterize the numbers $M_{i,n}$ and $M_{e,n}$ since

$$M_{i,n}=\sharp(\mathcal{R}_n\cap[0,1[)\text{ and }M_{e,n}=\sharp\left(\mathcal{R}_n\cap]1,\left(\sin\left(\frac{\pi}{n}\right)\right)^{-2}]\right).$$

The rest of this paper is organized as follows. Next section presents some properties of the geometric graph associated to the clique while the section after focuses mainly on the mappings J_n and $\tilde{J}_n$ which give the radii of the circles on which lie the intersection points of the graph $\mathcal{K}_n$. The next section is devoted to combinatorics inside the domain of J_n consisting in quadruplets (i,j,k,ℓ) of integers constrained by some specific inequalities. Afterwards, we use galoisian properties of $\mathbb{Q}(\zeta_{2n})$ in order to highlight the number of orbits associated to $\mathcal{K}_n$ and the numbers of vertices lying on those orbits. In the end, we conclude with open problems.

The Geometric Graph Associated to the Clique Arrangement

In this section, we determine the coordinates of the intersection points of the clique arrangement $\mathcal{K}_n$ as well as the modulii of these points. Next, we deal with symmetries and multiplicities. To define properly the vertices $z_{i,j,k,\ell}$, we may assume that $0\le i<j\le n-1,\ 0\le k<\ell\le n-1,\ i\le k$.

Lemma 1. We have $\{i,j\}\cap\{k,\ell\}\neq\emptyset\Leftrightarrow z_{i,j,k,\ell}\in\mathcal{C}_n$.

Proof. The implication $(\Rightarrow)$ is obvious. The converse implication relies on the following property resulting from a convexity argument: no three distinct vertices z_i, z_j and z_k of the regular n-gon are aligned. □

The intersection points $z_{i,j,k,\ell}$ are divided into two classes, depending on whether or not the indices $i+j$ and $k+\ell$ are equal to n or equivalently whether or not $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are vertical straight lines. Let us mention that when the integers i,j,k,ℓ are chosen in such a way that $i+j=k+\ell=n$ then the two straight-lines (z_i,z_j) and (z_k,z_ℓ) are parallel and disjoint and consequeltly, no intersection point occurs. The first class of intersection points is described as follows.

Lemma 2. We suppose that $i+j,k+\ell,n$ are three distinct integers. The parallax of the angular segment in the unit disk limited by O and the vertices $z_{i,j}$ and $z_{k,\ell}$, seen from $z_{i,j,k,\ell}$, is equal to $\theta = ij_+ - k\ell_+ = \left(i + j - (k+\ell)\right)\frac{\pi}{n}$. The cartesian coordinates of the intersection point $z_{i,j,k,\ell} = \mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}$ are equal to

$$x_{i,j,k,\ell} = \frac{\sin(ij_+)\cos(k\ell_-) - \cos(ij_-)\sin(k\ell_+)}{\sin(ij_+)\cos(k\ell_+) - \cos(ij_+)\sin(k\ell_+)} = \frac{\sin(ij_+)\cos(k\ell_-) - \cos(ij_-)\sin(k\ell_+)}{\sin(\theta)},$$

$$y_{i,j,k,\ell} = \frac{\cos(ij_-)\cos(k\ell_+) - \cos(ij_+)\cos(k\ell_-)}{\sin(ij_+)\cos(k\ell_+) - \cos(ij_+)\sin(k\ell_+)} = \frac{\cos(ij_-)\cos(k\ell_+) - \cos(ij_+)\cos(k\ell_-)}{\sin(\theta)},$$

while $\left|z_{i,j,k,\ell}\right|^2 = J_n(i,j,k,\ell)$.

Proof. Straightforward computations provide the equation of the straight line $\mathcal{D}_{i,j}$ passing through z_i and z_j ($i \neq j$ and $i + j \neq n$)

$$\mathcal{D}_{i,j}: y = -\cot(ij_+)x + \frac{\cos(ij_-)}{\sin(ij_+)}.$$

Solving the system of equations for $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ yields the cartesian coordinates of $z_{i,j,k,\ell}$. Squaring and adding these coordinates gives the formula for the square of the modulus. □

The second class of intersection points consists in the points $z_{i,j,k,\ell}$ where $i + j$ or $k + \ell$ is equal to n.

Lemma 3. Let us choose integers i,j,k,ℓ such that $i + j = n$ and $k + \ell \neq n$. Then the cartesian coordinates of the intersection point $\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}$ are equal to

$$x_{i,j,k,\ell} = \cos(ii_+), \qquad y_{i,j,k,\ell} = \frac{\cos(k\ell_-) - \cos(k\ell_+)\cos(ii_+)}{\sin(k\ell_+)},$$

while $\left|z_{i,j,k,\ell}\right|^2 = \tilde{J}_n(i,j,k,\ell)$. In the case where the integers i,j,k,ℓ are such that $i + j \neq n$ and $k + \ell = n$, $x_{i,j,k,\ell}$, $y_{i,j,k,\ell}$ and $\left|z_{i,j,k,\ell}\right|$ are obtained by replacing the quadruplet (i,j,k,ℓ) by (k,ℓ,i,j).

The proof is similar to the previous one by considering that this time $\mathcal{D}_{i,j}$ (or $\mathcal{D}_{k,\ell}$) is a vertical straight line.

Let us recall by the way that, given two straight lines with slopes s_1, s_2, the angle θ between these lines verifies the relationship $\tan\theta = \left|\frac{s_1 - s_2}{1 + s_1 s_2}\right|$. The slopes of $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are given by $s_1 = -\cot(ij_+)$ and $s_2 = -\cot(k\ell_+)$, and we have

$$\tan\theta = \left|\frac{\cot(ij_+) - \cot(k\ell_+)}{1 + \cot(ij_+)\cot(k\ell_+)}\right| = \left|\left(\cot(k\ell_+ - ij_+)\right)^{-1}\right|,$$

using the addition formula for cotangent. Hence, $\theta = \frac{\pi}{n}|k + \ell - i - j|$. This angle is the analog of the *parallax* in astronomy since it subtends any chord from z_i or z_j to z_k or z_ℓ. □

The upper bound for the number of intersection points N_n given in the introduction is nothing else than

$$N_n \leq q_n'' = n + \sharp\{(i,j,k,\ell), i < j, k < \ell, i < k, j \neq \ell, j \neq k, i + j - (k + \ell) \not\equiv 0 \bmod n\}.$$

These various inequalities and constraints forbid the intersection points to be over-counted.

Let us introduce the rotation ρ with center at origin and angle $\frac{2\pi}{n}$ defined as $\rho(z) = z\exp\left(\frac{2I\pi}{n}\right)$ in the plane $\mathbb{C} \simeq \mathbb{R}^2$. Obviously, one has $\rho^n = Id$,

$\rho(z_p) = z_{p+1}$ and $\rho^m(z_p) = z_{p+m}$ for all $m \in \mathbb{N}$ and all $p \in \{0, \dots, n-1\}$. We see also that $\rho(\mathcal{D}_{i,j}) = \mathcal{D}_{i+1,j+1}$ and that the vertex $z_{i,j,k,\ell}$ transforms as follows

$$\rho(z_{i,j,k,\ell}) = \rho(\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell}) = \mathcal{D}_{i+1,j+1} \cap \mathcal{D}_{k+1,\ell+1} = z_{i+1,j+1,k+1,\ell+1}.$$

As a consequence,

$$\begin{aligned} x_{i+1,j+1,k+1,\ell+1} &= x_{i,j,k,\ell}\cos\left(\frac{2\pi}{n}\right) - y_{i,j,k,\ell}\sin\left(\frac{2\pi}{n}\right), \\ y_{i+1,j+1,k+1,\ell+1} &= x_{i,j,k,\ell}\sin\left(\frac{2\pi}{n}\right) - y_{i,j,k,\ell}\cos\left(\frac{2\pi}{n}\right). \end{aligned}$$

More generally, $\rho^m(z_{i,j,k,\ell}) = z_{i+m,j+m,k+m,,\ell+m}$ for all $m \in \mathbb{N}$. In addition, symmetry considerations provide the following results

$$z_{i,j,k,\ell} = z_{k,\ell,i,j}, \qquad z_{i+n,j+n,k,\ell} = z_{i,j,k+n,\ell+n} = z_{i,j,k,\ell}.$$

The central symmetry holds only for even n and is contained in the following formula.

$$-z_{i,j,k,\ell} = z_{i+\frac{n}{2},j+\frac{n}{2},k,\ell} = z_{i,j,k+\frac{n}{2},\ell+\frac{n}{2}}.$$

So, according to rotations, we see that each orbit contains a multiple of n points. Let us arrange the orbits $\omega_1, \dots, \omega_{M_n}$ of $\mathcal{K}_n$ by increasing values of radius. For each p, the number of vertices of ω_p is a multiple $\nu_{n,p}n$ of n. So we get

$$N_{2n} = 1 + n \sum_{p=1}^{M_{2n}-1} \nu_{2n,p} \geq 1 + n(M_{2n} - 1)\,,$$

$$N_{2n+1} = n \sum_{p=1}^{M_{2n+1}} \nu_{2n+1,p} \geq nM_{2n+1}.$$

We give below, for $n = 8, 9, 10$, for the sake of conciseness, the cardinalities of the orbits of $\mathcal{K}_n$. To be cautious about the parity of n, we introduce the sequence Γ_n of lengths $M_{i,n} + 1 + M_{e,n}$ as follows

$$\Gamma_n = \begin{cases} \left(1, (n\nu_{n,p})_{p\in\{2,\dots,M_n\}}\right) & \text{for } n \text{ even} \\ (n\nu_{n,p})_{p\in\{1,\dots,M_n\}} & \text{for } n \text{ odd} \end{cases}.$$

In order to locate the n-gon $\mathcal{C}_n$, we underline its cardinality.

- Γ_8 =(1,8,8,8,16,8,<u>8</u>,8,16,16,8,8,16,16),
- Γ_9 =(9,9,9,18,9,18,9,18,18,9,<u>9</u> ,9,9,18,9,18,18,9,18,18,18,9,18,9,9),
- Γ_{10} = (1,10,10,10,10,20,10,20,10,10,20,20,10, <u>10</u>,10,20,20,10,20,20,10,20,10,20,20,20,10,20,10,20,20,20).

Proposition 1. For any integer $m \geq 1$, the sequences

$$(M_{mn})_{n\geq1}, (M_{e,mn})_{n\geq1}, (M_{i,mn})_{n\geq1}, (N_{mn})_{n\geq1}, (N_{e,mn})_{n\geq1}, (N_{i,mn})_{n\geq1}$$

are increasing.

Proof. Indeed, for any integer m, if $z_{i,j,k,\ell}$ is a vertex of $\mathcal{K}_n$ then this point coincides with the vertex $z_{mi,mj,mk,m\ell}$ of $\mathcal{K}_{mn}$. Moreover, the inclusion keeps the various properties of "being inside" or "being outside" the unit-circle. Similarly, each straight line $\mathcal{D}_{i,j}$ of $\mathcal{K}_n$ coincides with the straight line $\mathcal{D}_{mi,mj}$ of $\mathcal{K}_{mn}$. So the sets of vertices and of edges of $\mathcal{K}_n$ are included in the respective sets of vertices and of edges of $\mathcal{K}_{mn}$. □

In the following table we give for $3 \leq n \leq 10$, the values of cardinalities $N_n, N_{e,n}, N_{i,n}$, and the numbers of various orbits $M_n, M_{e,n}, M_{i,n}$.

Table 5. *Numbers of Intersection Points and Orbits*

n	N_n	$N_{e,n}$	$N_{i,n}$	M_n	$M_{e,n}$	$M_{i,n}$
3	3	0	0	1	0	0
4	5	0	1	2	0	1
5	15	5	5	3	1	1
6	37	18	13	6	2	3
7	91	49	35	10	5	4
8	145	88	49	14	7	6
9	333	198	126	25	14	10
10	471	300	161	32	18	13

Table 5 illustrates both Proposition 1 and previous formulas.

Elementary Properties of the Radius Mappings $J_n(i,j,k,\ell)$ and $\tilde{J}_n(i,j,k,\ell)$

In this section, i,j,k,ℓ denote indices lying in $[0, n-1]$ and such that $\sharp\{i,j,k,\ell\} \geq 3$. We have two formulas expressing symmetry by change of pairs of lines, $J_n(k,\ell,i,j) = J_n(i,j,k,\ell)$, $\tilde{J}_n(k,\ell,i,j) = \tilde{J}_n(i,j,k,\ell)$. We notice that, for any index m, we have the obvious relations

$$J_n(i,j,k,\ell) = J_n(i+m, j+m, k+m, \ell+m),$$

and

$$\tilde{J}_n(i,j,k,\ell) = \tilde{J}_n(i+m, j+m, k+m, \ell+m).$$

These relations mean that the functions J_n and $\tilde{J}_n$ are in a sense invariant by rotations. So one may assume $i = 0$ without loss of generality when studying those mappings. Using Lemmas 1 to 3, we get the identity

$$J_n(i+n, j+n, k, \ell) = J_n(i, j, k+n, \ell+n) = J_n(i,j,k,\ell).$$

The following property results from the examination of the explicit formulas given in previous section.

Lemma 4. Let $\{i,j,k,l\}$ such that the vertex $z_{i,j,k,\ell}$ exists. Then the four numbers $x_{i,j,k,\ell}$, $y_{i,j,k,\ell}$, $z_{i,j,k,\ell} = x_{i,j,k,\ell} + I y_{i,j,k,\ell}$, $|z_{i,j,k,\ell}|^2 = J_n(i,j,k,\ell)$ belong to the cyclotomic field $\mathbb{Q}(\zeta_{2n})$.

We emphasize on the fact that all the points z_m of the starting regular n-gon are in the cyclotomic field $\mathbb{Q}(\zeta_n)$, but however, the intersection points $z_{i,j,k,\ell} \notin \mathcal{C}_n$ lie in a larger field $\mathbb{Q}(\zeta_{2n})$.

When this is possible, in compliance with Galois's theorem, we may give explicit algebraic values of squared radii of orbits $\Xi_n = \left\{|z_{i+n,j+n,k,\ell}|^2\right\}$, which is done hereafter. Although Ξ_n being sets, we have sorted the values by increasing order. One has for instance $\Xi_3 = \{1\}$, $\Xi_5 = \left\{\frac{7-\sqrt{5}}{2}, 1, \frac{7+3\sqrt{5}}{2}\right\}$, $\Xi_6 = \left\{0, \frac{1}{3}, \frac{1}{2}, 1, 3, 7\right\}$ and by using old-school algebra, we see that Ξ_{10} contains

$$\{0,1,4\sqrt{5}+11,5+2\sqrt{5},\frac{1}{2}(7+3\sqrt{5}),\frac{1}{2}(7-3\sqrt{5}),$$
$$\frac{1}{2}(7+\sqrt{5}),4-\sqrt{5},4+\sqrt{5},\frac{1}{2}(3+\sqrt{5})\}.$$

For the sake of conciseness, the list has been volontarily shorten up but all the values do have explicit expressions. The method of Archimedes allows to compute explicitly and algebraically each radii of each set Ξ_{2^n} starting with $\Xi_4 = \{0,1\}$. For example Ξ_8 is equal to

$$\{0,3-2\sqrt{2},1-\frac{1}{2}\sqrt{2},\frac{1}{2},9-6\sqrt{2},1,1+\frac{1}{2}\sqrt{2},$$
$$5-2\sqrt{2},3,2+\sqrt{2},3+2\sqrt{2},5+2\sqrt{2},9+6\sqrt{2}\}.$$

But the formulas, although being known explicitly, become of a large complexity mainly due to Viète's formula.

Lemma 5. Let $w \in \mathbb{Q}(\zeta_N)$ be of modulus one, then $w = \zeta_N^m$ for some integer m, and conversely.

See the book by Washington (1997) for the proof. □

Lemma 6. We have $\sharp\{i,j,k,\ell\} = 3$ if and only if $J_n(i,j,k,\ell) = 1$ or $\tilde{J}_n(i,j,k,\ell) = 1$.

Proof. If a quadruplet is such that $\sharp\{i,j,k,\ell\} = 3$ with $i \neq j$ and $k \neq \ell$, it must be of the shape (i,j,i,ℓ) or (i,j,k,i) or (i,j,k,j) or (i,j,j,ℓ). Without loss of generality, we may suppose that $i = 0$ and that $k\ell = 0$. Up to a change of notation, the value of the function J_n for all four preceeding quadruplets is equal to 1 in view of the equation

$$\cos(x)^2 + \cos(y)^2 - 2\cos(x)\cos(y)\cos(\mathrm{x}-\mathrm{y}) = (\sin(x-y))^2.$$

Let us suppose that $\sharp\{i,j,k,\ell\} = 4$. We show that the requirement J_n (or $\tilde{J}_n$) is equal to 1 is contradictory. Indeed, by using Lemma 4, if J_n (or $\tilde{J}_n$) is equal to 1, the point $z_{i,j,k,\ell}$ in $\mathbb{Q}(\zeta_{2n})$ has modulus equal to 1, so it is a root of unity, i.e. $z_{i,j,k,\ell} = z_m$ for some index $m \in \{0,\dots,n-1\}$. So the index m is repeated twice which induces the contradiction. □

From now on in this section, we assume that $\sharp\{i,j,k,\ell\} = 4$.

Proposition 2. Let us suppose that $j - i = \ell - k$, difference that we denote t_1 and let us denote $t_2 = k + \ell - i - j$. Then

$$J_n(i,j,k,\ell) = \frac{1}{2}\left(\frac{\cos\left(\frac{t_1\pi}{n}\right)}{\cos\left(\frac{t_2\pi}{2n}\right)}\right)^2.$$

If η is such a value, so is $\frac{1}{4\eta}$. Lastly, the number of these values is $(n-1)(n-2)$.

Proof. The explicit computation of J_n relies on the duplication formulas for sine and cosine. Since $\sharp\{i,j,k,\ell\} = 4$, the equality $t_1 = \frac{1}{2}t_2$ does not occur; indeed, otherwise $j = k = i + t_1$. So the value of J_n may not be equal to $\frac{1}{2}$.

Let $\eta = J_n(i,j,k,\ell)$ with $j - i = \ell - k = t_1$. Since we have $t_2 = 2(k - i)$, t_2 must be even. Now,

$$\frac{1}{4\eta} = \frac{1}{2}\left(\frac{\cos\left(\frac{t_2\pi}{2n}\right)}{\cos\left(\frac{t_1\pi}{n}\right)}\right)^2 = J_n(i',j',k',\ell'),$$

where the indices i',j',k',ℓ' are chosen so that

$$j' - i' = \ell' - k' = \frac{t_2}{2}\ ,\ k' + \ell' - i' - j' = 2t_1.$$

We get $j' = i' + \frac{t_2}{2}$, $k' = i' + t_1$, $\ell' = i' + t_1 + \frac{t_2}{2}$. These integers are all four distinct and satisfy the hypothesis of the lemma, i.e. $j' - i' = \ell' - k'$. Lastly, the number of those values is equal to $(n-1)(n-2)$ since, we choose arbitrarily the integer $\frac{t_2}{2}$ in $[1, n-1]$ and next an integer $t_1 \in [1, n-1]$ distinct of $\frac{t_2}{2}$. □

The Combinatorics of Quadruplets of Indices of Intersection Points

We denote $Q_n \subset [0, n-1]^4$ the set of quadruplets of integers (i,j,k,ℓ), $0 \le i,j,k,\ell \le n-1$ with $i \ne j$, $k \ne \ell$. Hence Q_n is in bijection with the set of pairs of well-defined lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ being parallel or concurrent through some vertex of $\mathcal{K}_n$. The commutative group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts transitively on Q_n as follows: if (g, b) is in this group, then

$$b \cdot (i,j,k,\ell) = (k,\ell,i,j) \text{ and}$$
$$g \cdot (i,j,k,\ell) \equiv (i+g, j+g, k+g, \ell+g) \bmod n.$$

The group $\mathbb{Z}/2\mathbb{Z}$ acts as the identity on the set of vertices of the arrangement while $\mathbb{Z}/n\mathbb{Z}$ acts as the rotation group on this set of vertices.

Lemma 7. The direct image of $\tilde{J}_n$ is included in the direct image of J_n while being distinct.

Proof. Let $\eta \in \Im(\tilde{J}_n)$ then there exist i,j,k,ℓ with $j = n - i$ such that $\eta = \tilde{J}_n(i,j,k,\ell) = |z_{i,j,k,\ell}|^2$. We choose any integer p such that $2p \equiv 0 \bmod n$ and $2p + k + \ell \not\equiv 0 \bmod n$. In view of the properties of ρ, we have also $\eta = |z_{i+p,j+p,k+p,\ell+p}|^2 = J_n(i+p, j+p, k+p, \ell+p)$, the indices characterizing the quadruplet avoiding the case of vertical straight lines as explained in a previous section. So $\eta \in \Im(J_n)$.

Conversely, let $\eta \in \Im(J_n)$ and let us show that $\eta \in \Im(\tilde{J}_n)$ except in the case when n is even and $i + j$ and $k + \ell$ are odd.

In the three opposite cases, we may choose an integer m so that the quadruplet $(i+m, j+m, k+m, \ell+m)$ satisfies either $i + j + 2m \equiv 0 \bmod n$ or $k + \ell + 2m \equiv 0 \bmod n$. Indeed, $\eta = J_n(i,j,k,\ell)$ with $i + j \ne 0 \bmod n$ and $k + \ell \ne 0 \bmod n$. We choose $m = \frac{1}{2}(n - i - j)$ or $m = \frac{1}{2}(n - k - \ell)$ and we have $\eta = J_n(i+m, j+m, k+m, \ell+m) = \tilde{J}_n(i+m, j+m, k+m, \ell+m)$ with at least one of the two conditions $2m \ne n - k - \ell$ or $2m \ne n - i - j$. Let us consider now the fourth remaining case, that is to say the case when n is even and $i + j$ and $k + \ell$ are odd. No transformation of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, except the identity, preserves the value $J_n(i+p, j+p, k+p, \ell+p)$ and accordingly the radius $J_n(i+p, j+p, k+p, \ell+p)$ cannot be sent to a radius of the shape $\tilde{J}_n(i', j', k', \ell')$. □

We may synthetize the proof by saying that, without changing the value of $J_n(i,j,k,\ell)$ or $\tilde{J}_n(i,j,k,\ell)$, one may apply a transformation $(i,j,k,\ell) \to (i+p, j+p, k+p, \ell+p)$, so either to remove the condition $i+j \equiv 0 \bmod n$ or to impose it.

Let $Q_n^{(3)} \subset Q_n$ denote the subset of quadruplets such that $\#\{i,j,k,\ell\} = 3$. Orbits of Q_n are either contained in $Q_n^{(3)}$ or in $Q_n - Q_n^{(3)}$. In each orbit contained in $Q_n - Q_n^{(3)}$, one may find a representative such that $i < j$ and $k < \ell$ and $i < k$ and $j \neq \ell$. This allows us to define the set Q'_n of quadruplets (i,j,k,ℓ) verifying

$$0 \leq i < j \leq n-1, \;\; 0 \leq k < \ell \leq n-1, \;\; i < k, \;\; j \neq \ell.$$

Now, the orbits contained in $Q_n - Q_n^{(3)}$ are identified with the set of admissible intersection points $z_{i,j,k,\ell} \notin \mathcal{C}_n$, and of points that do not exist due to parallelism, since some of these quadruplets fail to define properly any intersection point because the underlying straight-lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ are parallel.

Let us denote Q''_n the subset of Q'_n of which the elements are $(i,j,k,\ell) \in Q'_n$ satisfying $i+j \neq k+\ell \bmod n$. This implies that $z_{i,j,k,\ell}$ exists, since in that case the parallax $|i+j-k-\ell|\frac{\pi}{n}$ is not flat nor equal to zero. When $(i,j,k,\ell) \in Q'_n$ the value of $|i+j-k-\ell|$ is less than $2n-2$, and thus the condition $i+j \neq k+\ell \bmod n$ is equivalent to $k+\ell-i-j \notin \{-n, 0, n\}$.

We define $q'_n = n + \#(Q'_n)$, and $q''_n = n + \#(Q''_n)$, both summands n counting the vertices of $\mathcal{C}_n$. In this section we will give explicit formulas for these two integer sequences. Note that the conditions in Q'_n are not necessary nor sufficient to define properly $z_{i,j,k,\ell}$, because the action of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ does not preserve these conditions, although it leaves invariant intersection points.

However, the conditions in Q'_n allow a crude estimation of the sequence (N_n) since we have:

Proposition 3. One has $\#(Q'_n) = \frac{1}{8}n(n^3 - 6n^2 + 11n + 2)$,

$$\#(Q''_n) = \frac{1}{16}n(2n^3 - 14n^2 - (-1)^n + 30n - 3), \text{ and } N_n \leq n + \sharp(Q''_n).$$

Proof. We first compute the cardinality of the set Q'_n. To do this, we split the cardinality as a sum of two triple sums of characteristic functions as follows

$$\#(Q'_n) = \left[\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=i+1}^{j-1} + \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n-1}\right]\#\{\ell, \ell \neq j, \ell > k\} = s_1 + s_2,$$

the triple sum being distributed over the characteristic functions and where we have set

$$s_1 = \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=i+1}^{j-1}(n-k-2)\,, s_2 = \sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n-1}(n-k-1).$$

Internal sums are easily computed:

$\sum_{k=i+1}^{j-1}(n-k-2)=\frac{1}{2}(2nj-j^2-3j-2ni-2n+i^2+5i+4)$ and
$\sum_{k=j+1}^{n-1}(n-k-1)=\frac{1}{2}(n^2-3n-2nj+j^2+3j+2)$.
So we obtain

$$s_1=\frac{1}{2}\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}(2nj-j^2-3j-2ni-2n+i^2+5i+4),$$

$$s_2=\frac{1}{2}\sum_{i=0}^{n-4}\sum_{j=i+1}^{n-1}(n^2-3n-2nj+j^2+3j+2).$$

Once again we compute internal sums, say s_1' and s_2',
$s_1'=\frac{1}{2}\sum_{j=i+1}^{n-1}(j^2-3j-2ij+i^2+3i+2)$,
$s_2'=\frac{1}{2}\sum_{j=i+1}^{n-1}(n^2-3n-2nj+j^2+3j+2)$.
We find that they are equal respectively to

$$s_1' = \frac{1}{3}(-3n^2i-6n^2+3i^2n+12ni+11n+n^3-i^3-6i^2-11i-6),$$

$$s_2' = \frac{1}{6}(n^3-6n^2+11n-3n^2i+12ni-11i-6+3i^2n-6i^2-i^3).$$

Now, adding those expressions to rule out the sum $s_1+s_2=\sum_{i=0}^{n-4}(s_1'+s_2')$ leads easily to $\#(Q_n')=\frac{1}{8}n(n^3-6n^2+11n+2)$.

The adaptation of these calculations to compute $\sharp(Q_n'')$ leads to intractable computations depending on the parity of n and we choose instead a distinct approach. To do this, we observe that the positive integers $q_n'-q_n''=\#(Q_n')-\#(Q_n'')$ enumerate admissible quadruplets (i,j,k,ℓ) in (Q_n') for which the straight-lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are parallel.

Let us consider the clique $\mathcal{K}_n$ associated to $\mathcal{C}_n$ with vertices set $E(\mathcal{K}_n)=\{z_m\}_{0\le m\le n-1}$. This is a geometric graph that we may convert in a digraph, i.e. a directed graph, in $2^{\frac{1}{2}n(n-1)}$ ways. Let us choose some orientation among them, giving rise to a transitive tournament. Let $\phi: E(\mathcal{K}_n)\to[0,n[$ be the 1-to-1 mapping sending each vertex z_m to its rank $\phi(m)$ in a complete tournament on the vertices, so that an edge (z_i,z_j) appears if and only if $\phi(z_i)<\phi(z_j)$.

We consider a regular N-gon with $N=\frac{1}{2}n(n-1)$ vertices, being pairs $\{z_i,z_j\}$, with indices $0\le i<j\le n-1$.

Let $E\left(\mathcal{K}_n^{(2)}\right)$ be this set of pairs $\{z_i,z_j\}$. A choice of the mapping ϕ corresponds to an unambiguously defined mapping $\psi: E\left(\mathcal{K}_n^{(2)}\right)\to[0,N[$ given by $\psi(z_i,z_j)\to\phi(z_i)+\phi(z_j)$. In order to define a tournament on the set $E\left(\mathcal{K}_n^{(2)}\right)$, we must exclude inconsistent quadruplets $\{\{z_i,z_j\},\{z_k,z_\ell\}\}$, for which $\psi(z_i,z_j)=\psi(z_k,z_\ell)$. So, inconsistent triplets, for which $\#\{z_i,z_j,z_k,z_\ell\}=3$, must be removed. This being, an ordering may be chosen on the vertices set, by setting $\{z_i,z_j\}\preccurlyeq\{z_k,z_\ell\}$ when $\psi(z_i,z_j)<\psi(z_k,z_\ell)$. This sequence corresponds

to the number of binary strings of length $n+1$ with exactly one pair of adjacent 0's and exactly two pairs of adjacent 1's, and constitutes the entry A080838 in Sloane (2023). It results therefore that the number of inconsistent quadruplets is equal to $\frac{1}{16}n(2n^2-8n+7+(-1)^n)$. With this preparation, one has $\#(Q_n'')=\frac{1}{16}n(2n^3-14n^2-(-1)^n+30n-3)$, the proof being nothing but substracting $\#(Q_n')-\#(Q_n'')$ from $\#(Q_n')$ computed earlier.
It remains to prove that $N_n \leq n+\sharp(Q_n'')$. We recast Lemma 6 as follows: Let (i,j,k,ℓ) so that $0 \leq i<j<n,\ i<k<\ell<n,\ k+\ell-i-j \notin \{-n,0,n\}$. Then

$$\sharp\{i,j,k,\ell\}=3 \Leftrightarrow \left|z_{i,j,k,\ell}\right|=1 \Leftrightarrow z_{i,j,k,\ell} \in \mathcal{C}_n \Leftrightarrow (i,j,k,\ell) \notin Q_n'.$$

Indeed, under the assumptions of Lemma 6, the intersection point $z_{i,j,k,\ell}$ exists. The equivalences between the three first statements are a consequence of Lemma 6 together with its proof while the equivalence between the fourth and the first statements is easy. Now, $\sharp(Q_n'')$ is an upper bound of $N_{e,n}+N_{i,n}$ in view of the contraposition. Adding the n points located on $\mathcal{C}_n$ we obtain the upper bound mentioned in the proposition. This ends the proof of the Proposition. □

The sequence $q_n'=n+\#(Q_n')$ begins as follows

Table 6. *Upper-bound for Number of Intersection Points*

n	3	4	5	6	7	8	9	10	11	12	13	14	…
q_n'	3	7	20	51	112	218	387	640	1001	1497	2158	3017	…

The numbers of lost points of intersection regarding to parallel straight lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ begins as

Table 7. *Upper-bound for Number of Lost Points due to Parallelism*

n	3	4	5	6	7	8	9	10	11	12	13	14	…
$q_n'-q_n''$	0	2	5	12	21	36	54	80	110	150	195	252	…

These tables allow to recover the tables given in the introduction. When we divide the numbers $q_n'-q_n''$ (except the very first one) by the number $n \geq 1$ of points in $\mathcal{C}_n$, one obtains a new interesting sequence of integers, namely

0, 0, 1, 2, 3, 4, 6, 8, 10, 12, 15, 18, 21, 24, 28,…

This last sequence is made of successive blocks of four increasing integers in arithmetic progression $(a, a+b, a+2b, a+3b)$, with a ratio b belonging itself to the arithmetic progression made by the integers $\mathbb{N}$ while the starting point a of each quadruplet belongs to a sequence of shifted triangular numbers.

Let us discuss now the multiplicities and the directions of parallelism of the clique arrangement. We refer to the works of Wetzel (1978) and Ryckelynck and Smoch (2023) for the background and definitions.

Proposition 4.

1. The multiplicity of the origin is 0 if n is odd and $\frac{n}{2}$ if n is even.

2. The number of quadruplets $(i,j,k,\ell) \in \mathcal{Q}_n$ such that $z_{i,j,k,\ell} = 0$ is 0 if n is odd and $\binom{\frac{n}{2}-1}{2}$ if n is even.
3. The multiplicity of $z_m \in \mathcal{C}_n$ is $n-1$.
4. The number of quadruplets $(i,j,k,\ell) \in \mathcal{Q}_n$ such that $z_{i,j,k,\ell} = z_m$ is $\binom{n-1}{2}$.
5. The number of directions for which more than one parallel occurs is equal to n.
6. The number of quadruplets $(i,j,k,\ell) \in \mathcal{Q}_n$ such that the lines $\mathcal{D}_{i,j}$ and $\mathcal{D}_{k,\ell}$ are parallel is equal to $\binom{n}{2}$.

Proof.
1. The multiplicity of the origin is the number of couples of straight lines $(\mathcal{D}_{i,j}, \mathcal{D}_{k,\ell})$ with $\mathcal{D}_{i,j} \cap \mathcal{D}_{k,\ell} = 0$. But when n is odd, if $z_m \in \mathcal{C}_n$ for some index $m \in \{0, \dots, n-1\}$ then $-z_m \notin \mathcal{C}_n$ since $(-z_m)^n = (-1)^n = -1$. When n is even, all the $\frac{n}{2}$ straight lines $\mathcal{D}_{i,\mathrm{i}+\frac{n}{2}}$ are concurrent through the origin.
2. The number of quadruplets (i,j,k,ℓ) such that $z_{i,j,k,\ell} = 0$, i.e. such that z_i and z_j (respectively z_k and z_ℓ) are symmetric with respect to the origin, is the number of quadruplets of the shape $\left(i, i+\frac{n}{2}, k, k+\frac{n}{2}\right)$, with $0 \leq i < k < \frac{n}{2}$ whence the formula.
3. For all $m \in \{0, \dots, n-1\}$, the $n-1$ straight-lines $\mathcal{D}_{i,m}$ are concurrent through the point z_m.
4. The quadruplets we are looking for are congruent modulo $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ to quadruplets of the shape (m,i,m,j) where $0 \leq i < j < n$ and $i \neq m$, $j \neq m$ whence the result.
5. Let us fix an integer i. All quadruplets of the shape $(i, i+1, i-k, i+1+k)$ are such that the straight lines $\mathcal{D}_{i,i+1}$ and $\mathcal{D}_{i-k,i+1+k}$. Let us consider the two mid-points $\frac{1}{2}(z_i + z_{i+1})$ and $\frac{1}{2}(z_{i-k} + z_{i+1+k})$. The line passing through these mid-points is the bisection of the segments $[z_i, z_{i+1}]$ and $[z_{i-k}, z_{i+1+k}]$ and thus is perpendicular to these segments. So the lines supporting them are parallel.
6. Since we have $i \in \{0, \dots, n-1\}$ and $k \in \{0, \dots, (n-1)/2\}$, the result holds. □

The Galoisian Properties of the Values of Squared Moduli $\boldsymbol{J_n(i,j,k,\ell)}$

The cyclotomic field $\mathbb{Q}(\zeta_n)$ is a galoisian extension of the rationals $\mathbb{Q}$ of degree $\varphi(n)$ with ring of integers $\mathbb{Z}[\zeta_n]$ and with Galois's group $(\mathbb{Z}/n\mathbb{Z})^*$ of which the operation on $\mathbb{Q}(\zeta_n)$ is done through exponentiation $\zeta_n \to \zeta_n^g$. We denote as usual by $\Phi_n(z) \in \mathbb{Z}[z]$ the cyclotomic polynomial of index n and degree $\varphi(n)$, which is the minimal polynomial of ζ_n and whose roots are all primitive roots of unity.

In this section we give formulas for $J_n(i,j,k,\ell)$ and $J_n'(i,j,k,\ell)$ in the field $\mathbb{Q}(\zeta_{2n})$. In other words we concentrate on formulas such as those given in the following result:

Proposition 5. For all integers $n \geq 1$ there exists a smaller integer b_n such that for all $(i,j,k,\ell) \in Q''_n$, one may find a vector of integers (λ_m) such that one has the formula $J_n(i,j,k,\ell) = \frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1} \lambda_m \, \zeta_n^m$.

Proof. Let us fix n and put $\zeta = \zeta_{2n}$. For all k, we have $\cos\left(k\frac{\pi}{n}\right) = \cos\left(k\frac{2\pi}{2n}\right) = \frac{1}{2}(\zeta^k + \zeta^{-k}) = \frac{1}{2}(\zeta^k + \zeta^{2n-k})$. So, $\cos^2\left(k\frac{\pi}{n}\right) = \frac{1}{4}(\zeta^{2k} + \zeta^{2n-2k} + 2)$.
Hence if $(i,j,k,\ell) \in Q''_n$, and if we set $p = j - i, q = \ell - k, r = k + \ell - i - j$, (with $p + q + r$ an even positive integer) we get

$$J_n(i,j,k,\ell) = \frac{\cos^2\left(p\frac{2\pi}{2n}\right)+\cos^2\left(q\frac{2\pi}{2n}\right)-2\cos\left(p\frac{2\pi}{2n}\right)\cos\left(q\frac{2\pi}{2n}\right)\cos\left(r\frac{2\pi}{2n}\right)}{1-\cos^2\left(r\frac{2\pi}{2n}\right)}$$

and so $J_n(i,j,k,\ell)$ is equal to

$$\frac{4 + \zeta^{2p} + \zeta^{2n-2p} + \zeta^{2q} + \zeta^{2n-2q} - (\zeta^p + \zeta^{2n-p})(\zeta^q + \zeta^{2n-q})(\zeta^r + \zeta^{2n-r})}{2 - \zeta^{2r} - \zeta^{2n-2r}}.$$

This is the evaluation at $z = \zeta_{2n}$ of the rational function N/D where we have set $N = 4 + z^{2p} + z^{-2p} + z^{2q} + z^{-2q} - z^{r+p+q} - z^{r+p-q} - z^{r-p+q} - z^{r-p-q} - z^{-r+p+q} - z^{-r+p-q} - z^{-r-p+q} - z^{-r-p-q}$, and $D = 2 - z^{2r} - z^{2n-2r}$.
Let us fix $(i,j,k,\ell) \in Q''_n$ and let us denote p, q, r the positive integers

$$p = j - i \,,\; q = \ell - k \,,\; r = k + \ell - i - j.$$

We noting that $p + q + r$ is even. We introduce integers $(\lambda_m)_{0 \leq m < \varphi(2n)}$ and a common denominator b_n (all these integers being at the time indetermined) and we define the polynomial with rational coefficients

$$P(z) = \frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1} \lambda_m \, z^m.$$

With p, q, r, b_n being fixed, let us show how to characterize in a unique way the sequence (λ_m) so that the two polynomials
$b_n(4 + z^{2p} + z^{-2p} + z^{2q} + z^{-2q} - z^{r+p+q} - z^{r+p-q} - z^{r-p+q} - z^{r-p-q} - z^{-r+p+q} - z^{-r+p-q} - z^{-r-p+q} - z^{-r-p-q})$,
and $(2 - z^{2r} - z^{2n-2r})\sum_{m=0}^{\varphi(2n)-1} \lambda_m \, z^m$, are congruent modulo the ideal generated in the Laurent algebra of rational functions $\mathbb{Q}[z, z^{-1}]$ by the cyclotomic polynomial $\Phi_{2n}(z)$.
To obtain a matricial system for the vector $\frac{1}{b_n}(\lambda_m)$, we compute the remainder of the euclidean division of the numerator of $N(z)/D(z) - P(z)$ by the cyclotomic polynomial $\Phi_{2n}(z) \in \mathbb{Z}[z]$, and require vanishing of all its coefficients. This determines uniquely the coefficients $\frac{1}{b_n}(\lambda_m)$. Now, the computation of b_n is nothing but searching the g.c.d. of all denominators of all characteristic vectors. □

The following corollary gives an application to Chebyshev polynomials of the first kind.
Proposition 6. For any integer $n \geq 3$ and (p,q,r) with $p + q + r$ even, if (λ_m) is defined as in Proposition 5, the polynomial

$$b_n\left(T_p(x)+T_q(x)-2T_p(x)T_q(x)T_r(x)\right)-(1-T_r(x)^2)\sum_{m=0}^{\varphi(2n)-1}\lambda_m\,x^m$$

is a multiple of the cyclotomic polynomial $\Phi_n(x)$.

Proof. Let us choose any quadruplet (i,j,k,ℓ) such that Proposition 5 holds. This is possible since $p+q+r$ is even. Since we have $T_k(\cos\theta)=\cos(k\theta)$ for any integer k and any real θ , we get

$$J_n(i,j,k,\ell)=\frac{T_p(\alpha)+T_q(\alpha)-2T_p(\alpha)T_q(\alpha)T_r(\alpha)}{1-T_r(\alpha)^2}$$

where $\alpha=\cos\left(\frac{\pi}{n}\right)$.

The rational fraction $\left(T_p(x)+T_q(x)-2T_p(x)T_q(x)T_r(x)\right)/\left(1-T_r^2(x)\right)$ and the polynomial $P(x)=\frac{1}{b_n}\sum_{m=0}^{\varphi(2n)-1}\lambda_m\,x^m$ agree at the point $x=\alpha$ in view of the previous proposition. So the polynomial in the claim is a multiple of the minimal polynomial of α. □

Remark 1. To compute the lattice vectors λ, one may devise the following algorithm. First, obtain g.c.d. between the numerator and the denominator of the fraction occuring in Proposition 6, next write the long division of this fraction, and at last, write a Bezout identity involving Chebyshev and cyclotomic polynomials. In this respect, we may refer to the work of Rayes et al. (2005) for properties like Bezout identities, resultants, g.c.d., for the family of polynomials (T_k). Nevertheless, the extension of those results to the required family of polynomials used in the previous procedure is very complicated to handle.

We continue this section with the study of the action of the Galois's group of $\mathbb{Q}(\zeta_{2n})$ on the sets Q'_n and Q''_n. If σ is an automorphism of $\mathbb{Q}(\zeta_{2n})$ then there exists an integer g prime to $2n$ such that one has $\sigma(\zeta_{2n})=\zeta_{2n}^{g}$ and conversely. Thus we have an isomorphism of groups $\sigma\colon g\to\sigma_g$ and we get $\sigma_g\left(\zeta_{2n}^{h}\right)=\zeta_{2n}^{g^h}$ for all integer $h\geq 0$. The following proposition is intended to transfer the action of the Galois group on the field to an action on the sets of admissible quadruplets.

Proposition 7. The Galois group $(\mathbb{Z}/2n\mathbb{Z})^*$ of $\mathbb{Q}(\zeta_{2n})$ acts faithfully on Q_n as follows. Let $g\in(\mathbb{Z}/2n\mathbb{Z})^*$ and $(i,j,k,\ell)\in Q_n$ then $\sigma_g(i,j,k,\ell)\equiv(i',j',k',\ell')$ modulo $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$ with

$$i'\equiv i\ ,\ j'-i'\equiv g(j-i)\ ,\ \ell'-k'\equiv g(\ell-k)\ ,$$
$$k'+\ell'-i'-j'\equiv g(k+\ell-i-j)\ \text{mod}\ n.$$

Each transformation σ_g preserves Q'_n and Q''_n. The orbits under the action of the Galois group have cardinalities divisors of $\varphi(2n)$. The conjugates of values of J_n (resp. $\tilde{J}_n$) are themselves values of J_n (resp. $\tilde{J}_n$). The mappings J_n and $\tilde{J}_n$ are contravariant with respect to these actions, i.e., $\sigma\circ J_n=J_n\circ\sigma$ for all $\sigma\in(\mathbb{Z}/2n\mathbb{Z})^*$. The Galois group acts as permutation group on the set of lattice vectors of radii cyclotomic representations.

Proof. First, we prove that the equations given in Proposition 7 determines without any ambiguity the quadruplet (i, j, k, ℓ). If we replace congruence modulo n by stricto sensu equalities, and if we use the triplet (p, q, r), the transformation becomes $p' = gp$, $q' = gq$, $r' = gr$ with $p' + q' + r' = g(p + q + r)$ even. In this way, we obtain the solution as

$$(i', j', k', \ell') \equiv (i, i + gp, i + g(k - i), i + g(k - i) + gq),$$

modulo $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.. Obviously, σ_g preserves by definition the set Q'_n and since when $r \not\equiv 0 \mod n$, we have also $gr \not\equiv 0 \mod n$, we see also that σ_g preserves also Q''_n in view of Gauss lemma. The formulas of Proposition 7 give rise to an equivalence relation since reflexivity occurs by choosing $g = 1$, symmetry holds by using g' such that $gg' \equiv 1 \mod 2n$, and transitivity holds with $g'' = gg'$ which is prime to $2n$. By Lagrange's theorem on finite groups, each equivalence class has cardinality being a divisor of the order of $(\mathbb{Z}/2n\mathbb{Z})^*$ which is $\varphi(2n)$. The previous formulas show that orbits have maximal cardinalities $\varphi(2n)$ when $j - i$, $k - i$ and $\ell - k$ are prime to n .

Let $g \in (\mathbb{Z}/2n\mathbb{Z})^*$ to which we associate an automorphism σ in the Galois group. For any quadruplet (i, j, k, ℓ) we may compute the conjugate $\sigma\big(J_n(i, j, k, \ell)\big)$ as follows. Let us introduce the three integers

$$p = j - i, q = \ell - k, r = k + \ell - i - j,$$

and the two polynomials:

$$w' = \zeta^{2gp} + \zeta^{2gn-2gp} + \zeta^{2gq} + \zeta^{2gn-2gq},$$

$$w'' = (\zeta^{gp} + \zeta^{2gn-gp})(\zeta^{gq} + \zeta^{2gn-gq})(\zeta^{gr} + \zeta^{2gn-gr}).$$

Then one obtains

$$\sigma\big(J_n(i, j, k, \ell)\big) = \frac{4 + \mathrm{w}' - \mathrm{w}''}{2 - \zeta^{2gr} - \zeta^{2gn-2gr}}$$

Expanding numerator and denominator shows that

$$\sigma\big(J_n(i, j, k, \ell)\big) = J_n(i', j', k', \ell'),$$

where the quadruplet (i', j', k', ℓ') is given by equations in Proposition 7. Now we deal with lattice vectors of radii cyclotomic representations. If $J_n(i, j, k, \ell) = \frac{1}{b_n}\sum_{m=0}^{n-1} \lambda_m \zeta^m$, we obtain $\sigma\big(J_n(i, j, k, \ell)\big) = \frac{1}{b_n}\sum_{m=0}^{n-1} \lambda_m \zeta^{gm}$. This implies that the lattice vectors of J_n and $\sigma \circ J_n$ are permuted vectors in $\mathbb{Z}^{\varphi(2n)}$. The remainder of the proof is straightforward. □

Proposition 8. The number of orbits with rational value of $J_n(i, j, k, \ell)$ is at least equal to the remainder of euclidean division of N_n by $\varphi(n)$.

Proof. Let us recall that the field $\mathbb{Q}(\zeta_{2n})$ is a galoisian extension of $\mathbb{Q}$, and, for any $\sigma \in (\mathbb{Z}/2n\mathbb{Z})^*$, σ generates $(\mathbb{Z}/2n\mathbb{Z})^*$ so σ has no fixed point except rational numbers. Otherwise, the subfield of invariant of σ won't be equal to $\mathbb{Q}$, contrarily to Galois's theorem. So, for any $(i, j, k, \ell) \in Q''_n$, $J_n(i, j, k, \ell)$ is irrational if and only if the equivalence class of $J_n(i, j, k, \ell)$ modulo the Galois's group has cardinality $\varphi(2n)$, or, translated in the action of $(\mathbb{Z}/2n\mathbb{Z})^*$ over Q''_n, the quadruplet (i, j, k, ℓ) spans an orbit of cardinality $\varphi(2n)$. Similarly, rational values of $J_n(i, j, k, \ell)$ give rise to distinct equivalence class in Q''_n modulo

$(\mathbb{Z}/2n\mathbb{Z})^*$ with cardinality 1. So, if $M_{r,n}$ (respectively $M_{s,n}$) denotes the number of rational (respectively irrational, with surds) values of $J_n(i,j,k,\ell)$, one has $\sharp(Q_n'') = M_{s,n}\varphi(2n) + M_{r,n}$. This implies our statement. □

Now, we conclude with concrete situations giving rise to explicit cyclotomic representations of the square radii. For the sake of brevity, we give only three examples, for n equal to 7, 8, 9.

Example 1. Let us consider the case $n = 7$ where $\varphi(14) = 6$ and $\Phi_{14}(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$. Let $\zeta = \zeta_{14}$ be a primitive root of unity of order 14, satisfying $\Phi_{14}(\zeta) = 0$. Consider for instance the radius
$j = -2(\zeta^4 - 2\zeta^3 + 3\zeta^2 - 2\zeta + 1) \cdot \zeta^{-1} \cdot (\zeta^2 - 2\zeta + 1)^{-1} \simeq 6.493959$.
Then the equations for the sequence $\lambda = (\lambda_5, \dots, \lambda_0)$ are
$\lambda_0 + \lambda_3 - 2\lambda_4 = 4, \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 = 0, -\lambda_3 + \lambda_4 + \lambda_5 = -2,$
$\lambda_1 - 2\lambda_2 + 2\lambda_4 - \lambda_5 = -2, -2\lambda_0 + \lambda_1 - \lambda_3 + 2\lambda_4 - \lambda_5 = -6,$
$\lambda_0 - 2\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4 + \lambda_5 = 4.$
Solving we find what we call the characteristic vector $\lambda = (\lambda_6, \dots, \lambda_0) \in \mathbb{Z}^6$ of the radius j, and in this case it is precisely $(\lambda) = (-2,0,0,2,0,4)$. More generally, computing with Maple, we obtain $b_7 = 1$ and the following representations of the various squares of radii starting with $j_0 = 1$. Then we list them by increasing height, that is to say by increasing sum of absolute values of the coefficients of the representation. The height is a good and simple test to recognize that algebraic values are conjugates since in that case we deal with two vectors (λ_m) cyclically permuted and thus of same height.
For the following pairs of square of radii, we see that the components share the same integer vector (λ_m) and so are congruent by the automorphism $\sigma: \zeta_{14} \to \zeta_{14}^5$, with $g = 5, j_6 = 2\zeta^4 - 2\zeta^3 + 4, j_7 = -2\zeta^5 + 2\zeta^2 + 4$, with height 8. Of height 9 is $j_1 = 2\zeta^5 + \zeta^4 - \zeta^3 - 2\zeta^2 + 3$. Of height 10 are $j_6 = 2\zeta^5 - 2\zeta^4 + 2\zeta^3 - 2\zeta^2 + 2$ and $j_2 = \zeta^5 - 3\zeta^4 + 3\zeta^3 - \zeta^2 + 2$, but they are not conjugates for the vectors (λ_m) are not cyclically permuted. Of height 15 is $j_4 = -3\zeta^5 + 2\zeta^4 - 2\zeta^3 + 3\zeta^2 + 5$. Of height 20 now is $j_7 = 3\zeta^5 + 4\zeta^4 - 4\zeta^3 - 3\zeta^2 + 6$. The sequence ends with two squares of raddi with height 24, $j_8 = 4\zeta^5 - 7\zeta^4 + 7\zeta^3 - 4\zeta^2 + 2$, and with height 29, $j_9 = -7\zeta^5 + 3\zeta^4 - 3\zeta^3 + 7\zeta^2 + 9$.

Example 2. Now we deal with the case $n = 9$ where $\varphi(9) = 6$ and $\Phi_{18}(z) = z^6 - z^3 + 1$. Let $\zeta = \zeta_{18}$ be a primitive root of unity of order 18, satisfying $\Phi_{18}(\zeta) = 0$. Computing with Maple, we obtain $b_9 = 12$ and we may obtain the cyclotomic representations of the all squares of radii each multiplied by 12. Then we see that the distribution of orbits is as follows. First we get seven isolated such circles, with rational values of square of radii and of increasing heights from 12 to 54 with orbits having each 9 vertices. Six of these orbits are inside the regular nonagon $\mathcal{C}_9$. For heights greater than 12, we obtain six further orbits. First of them, is of height 20 the radius $-4\zeta^5 + 4\zeta^4 + 12$, is of height 24 the radius $-4\zeta^5 + 4\zeta^2 + 4\zeta + 12$, is of height 32 the radius of an internal orbit given by $8\zeta^5 - 4\zeta^4 - 4\zeta^2 - 4\zeta + 12$, is of height 36 the radius $-4\zeta^5 - 4\zeta^4 + 8\zeta^2 + 8\zeta + 12$ is of height 48 the radius $-12\zeta^5 + 12\zeta^4 + 24$, and lastly, is of height 54 the radius $-12\zeta^5+3\zeta^4+9\zeta^2+9\zeta+21$. It is quite interesting to note that the radius of the six external orbits are increasing w.r.t. ordinary ordering. Next come two internal orbits with conjugate and distinct radii $12\zeta^4-12\zeta^2-12\zeta+24$, $12\zeta^5-12\zeta^2-12\zeta+24$ of height 60, and then three other such conjugate external orbits with squared radii having lattice vectors defined by

$$(-12,24,-12,-12,48),(24,-12,-12,-12,48),(-24,12,12,12,48),$$

all having height 108. At last, we find thirteen other values, that we may list by increasing value of the height 120, 168, 192, 264, 264, 300, 303, 366, 408, 768, 852, 1020, 1164 of the square radii. Since heights are distinct, these 13 orbits are non conjugate, and thus all these quantitites are rational numbers. Among the orbits, six are internal, seven are external, and the ordering with numerical order is not respected.

Example 3. The last example is given by $n=8$ where $\varphi(16)=8$ and $\Phi_{16}(z)=z^8+1$. This example is far more complicated. Let $\zeta=\zeta_{16}$ be a primitive root of unity of order 16, satisfying $\Phi_{16}(\zeta)=0$. We have $b_8=226$. We do not here provide the whole list of the sixteen squares of radii, but instead a few comments. If these squared radii are each multiplied by 226, the height are increasing and form the sequence:
0, 113, 226, 452, 452, 678, 886, 886, 886, 904, 904, 1338, 1338, 1808, 1808, 2658. In this way we obtain three sets of conjugate circles, one of cardinal 3 and two others of cardinal 2. When two vectors $(\lambda_7,\dots,\lambda_0)\in\mathbb{Z}^8$ appear and are obviously permutations one of the other, then these lattice vectors are conjugate through a galoisian morphism. This situation holds on six orbits forming three group of two orbits, defined help to the characteristic vectors $(\lambda_7,\dots,\lambda_0)\in\mathbb{Z}^8$:

$$(0,0,0,0,113,0,-113,226),(0,0,0,0,-113,0,113,226),$$
$$(0,0,0,0,-226,0,226,452),(0,0,0,0,226,0,-226,452),$$
$$(0,113,0,-226,-452,113,452,452),(0,113,0,-226,452,113,-452,452).$$

Orbits with cyclotomic vectors $(\lambda_7,\dots,\lambda_0)\in\mathbb{Z}^8$ where small changes of signs are done appear also by pairs and are not conjugate. This is the case of the three pairs

$$(-68,-8,132,-144,116,120,-172,578),$$
$$(68,-8,-132,-144,-116,120,172,578),$$
$$(204,-24,-396,-432,-348,360,516,378),$$
$$(-204,-24,396,-432,348,360,-516,378),$$
$$(68,-8,-132,-144,-116,120,172,126),$$
$$(-68,-8,132,-144,116,120,-172,126),$$

and they constitute strange gifts of the computation. We note also that the height takes three times the value 886 without giving rise to three conjugate orbits, only two of them being conjugates. This being so, we may analyze in the case $n=8$ the geometry of this clique arrangement help to the explicit values:

$$0,1,4\pm\sqrt{5},5\pm2\sqrt{5},\frac{1}{2}\pm\frac{1}{10}\sqrt{5},\frac{3}{2}\pm\frac{1}{2}\sqrt{5},$$
$$\frac{7}{2}\pm\frac{3}{2}\sqrt{5},\frac{7}{2}+\frac{1}{2}\sqrt{5},6+\sqrt{5},11+4\sqrt{5},\frac{3}{8}-\frac{1}{8}\sqrt{5}.$$

We observe in this case the appearance of radii of which some conjugates are removed. This is easily understood and arise because some orbits of Q_8' have non void intersections together with Q_8'' and $Q_8' - Q_8''$.

Remark 2. Let $\xi_n = 2\cos\left(\frac{2\pi}{n}\right) = \zeta_n + \zeta_n^{-1}$ and let $\mathbb{Q}(\xi_n) = \mathbb{Q}(\zeta_n) \cap \mathbb{R}$ be the maximal real subfield contained in the field $\mathbb{Q}(\zeta_n)$ and generated by ζ_n. The minimal polynomial $\Psi_n(z) \in \mathbb{Z}[z]$ of ξ_n over the rationals has been explicitly computed by Lehmer (1933). For $n \geq 3$, it is a non-zero monic polynomial of degree $(\varphi(2n) - 1)/2$ with integer coefficients. The preceeding formulas for $x_{i,j,k,\ell}$, $y_{i,j,k,\ell}$, $J_n(i,j,k,\ell)$, $J_n'(i,j,k,\ell)$, in both cases of Lemma 1 and Lemma 2, show that all these quantities are algebraic numbers belonging to $\mathbb{Q}(\xi_{2n})$. As a consequence, they may be written under the form $\frac{1}{b_n}\sum_{m=0}^{(\varphi(2n)-1)/2} a_m \, {\xi_n}^m$, for convenient and uniquely defined integers a_m, depending on i, j, k, ℓ and b_n depending only on n.

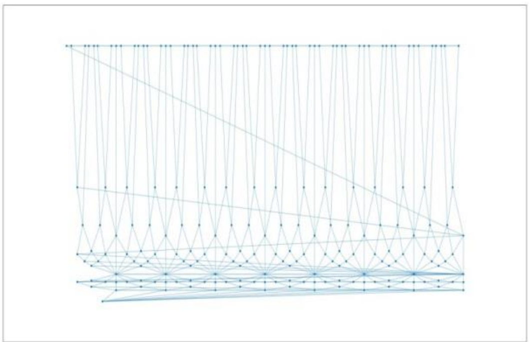
Since dealing with the non-monic polynomial $2^{\frac{1}{2}\varphi(2n)}\Psi_{2n}(z/2) \in \mathbb{Z}[z]$ closely related to Lehmer's polynomial is far from being easy, we do not take in account the reality of algebraic numbers.

Open Problems

In this section we conclude this work with some perspectives.

We may compare the clique arrangement $\mathcal{K}_n$, to which this paper is devoted, and the cyclotomic arrangement $\mathcal{R}_n$, see Ryckelynck and Smoch (2003), since their construction relies on $\mathcal{C}_n$. But the analogy ends there since geometrical, topological and combinatorial properties of $\mathcal{K}_n$ are far more complicated. This being so, we suggest as a first open problem the enumeration of the polygonal compact or non-compact chambers of $\mathbb{R}^2 - \mathcal{K}_n$ together with the description as it has been done for $\mathcal{R}_n$. In our opinion, this problem is the main one. Even if Poonen and Rubinstein (1998) have provided formula for the number of regions included in the unit disk $|z| \leq 1$, they did not characterize their geometry nor than the number of regions outside the disk.

Figure 2. *Polar Representation of the Geometric Graph of the Clique for $n = 8$*



It is quite surprising that the chambers of the cyclotomic arrangement $\mathcal{R}_n$ are of simple shapes, triangles or quadrilaterals or pentagons, compact or non compact. Experimentation with matlab shows that polygonal compact chambers of $\mathcal{K}_n$ with 6, 7 or 8 sides appear for examples with n smaller than 13. We propose as a difficult problem to prove that any number k of sides is admissible, i.e., for each integer k, there exists an integer n such that $\mathcal{K}_n$ admits at least one polygonal chamber with k sides.

Let us discuss one important step in the preceding program linked to the connectivity of the geometric graph $\mathcal{K}_n$. We must deal at first with the adjacency relation in the set of vertices in this geometric graph. If $z_{i,j,k,\ell}$ and $z_{i',j',k',\ell'}$ denote two intersection points, let us consider their connectivity $z_{i,j,k,\ell} \sim z_{i',j',k',\ell'}$. The straight line joining these two intersection points must coincide necessarily with two of the four straight lines $\mathcal{D}_{i,j}$, $\mathcal{D}_{i',j'}$, $\mathcal{D}_{k,\ell}$ or $\mathcal{D}_{k',\ell'}$ so we have for instance $(i,j) = (i',j')$. Next, the abscissas of $z_{i,j,k,\ell}$ and $z_{i',j',k',\ell'}$ along to $\mathcal{D}_{i,j}$ must be close one to the other in the sense that no other abscissa of $z_{i,j,k',\ell'}$ lies between the two previous ones.

Experimentations with $\mathcal{K}_n$ with $n = 3, \ldots, 12$, have led us to find explicit boolean adjacency matrices $\mu_n \in \mathcal{M}_{N_n}$ of a shape, in a way, similar to the adjacency matrix that we presented in the case of $\mathcal{R}_n$. We propose as an open problem to find and prove such adjacency matrices. We suggest that the generation of μ_n relies on a lexicographic ordering of the vertices using first increasing values of the radii and next, increasing values of the polar angle along the orbits. To obtain this lexicographic order, one must convert $\mathcal{K}_n$ in a lattice in the euclidean plane by using polar coordinates as depicted in the figure above.

A third interesting problem concerns the computation of the multiplicities of the intersection points. To do that, one must solve pair of trigonometric equations $x_{i,j,k,\ell} = x_0$ and $y_{i,j,k,\ell} = y_0$ and convert them into diophantine ones. This seems a very difficult problem.

References

Bol G (1936) Beantwoording van prijsvraag no. 17. (Answer to competition no. 17.) *Nieuw Archief voor Wiskunde* (18): 14–66.

Harborth H (1969a) Diagonalen im regularen n-Eck. (Diagonals in regular n-disk.) *Elem. Math*. (24): 104–109.

Harborth H (1969b) Number of intersections of diagonals in regular n -gons, Combinatorial Structures and their Applications. In *Proceedings of Calgary International Conference*, 151–153. Calgary, Alta.

Lehmer DH (1933) A note on trigonometric algebraic numbers. *American Mathematical Monthly* 40(3): 165–166.

Poonen B, Rubinstein M (1998) The number of intersection points made by the diagonals of a regular polygon. *SIAM Journal on Discrete Mathematics* 11(1): 135–156.

Rayes MO, Trevisan V, Wang PS (2005) Factorization properties of Chebyshev polynomials. *Computers and Mathematics with Applications* 50(8): 1231–1240.

Ryckelynck P, Smoch L (2023) On cyclotomic arrangements of lines in the plane. *International Journal of Geometry* 12(2): 70–88.
Rigby JF (1980) Multiple intersections of diagonals of regular polygons, and related topics. *Geometriae Dedicata* 9(Jun): 207–238.
Sloane N (2023) *The on-line encyclopedia of integer sequences*. The OEIS Foundation Inc.
Steinhaus H (1958) Problem 225. *Colloq. Math*. 5.
Steinhaus H (1983) *Mathematical snapshots*. Oxford University Press.
Tripp CE (1975) Adventitious angles. *Math. Gazette* (59): 98–106.
Washington LC (1997) *Introduction to cyclotomic fields*. Springer-Verlag, G.T.M. 83.
Wetzel JE (1978) On the division of the plane by lines. *The American Mathematical Monthly* 85(8): 647–656.